BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

SÉRIE DES SCIENCES
MATHÉMATIQUES, ASTRONOMIQUES ET PHYSIQUES

Volume VIII Numéro 4

VARSOVIE 1960

Le Bulletin de l'Académie Polonaise des Sciences est publié en Séries.

La présente Série comprend des communications du domaine de la mathématique, astronomie et physique

Lesdites sciences sont du domaine de la Troisième Classe de l'Académie Polonaise des Sciences.

La présente Série paraît mensuellement.

Les communications destinées à être publiées dans le Bulletin sont présentées à la Rédaction du Bulletin par les membres de l'Académie.

Adresse de la Rédaction: Pałac Kultury i Nauki, Varsovie

BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

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VOLUME VIII
NUMERO 4

VARSOVIE 1960

PRINTED IN POLAND

PAŃSTWOWE WYDAWNICTWO NAUKOWE, W-WA. MIODOWA 10

Nakład 1110+150 egz.	*Rękopis dostarczono 5.III.1960*
Ark. wyd. 4,75, ark. druk. 4.375+0,25	*Podpisano do druku 10.VII.1960*
Papier druk. sat. 80 g kl. III	*Druk ukończono w lipcu 1960*
Format B5, 70×100 cm	*Zam. prod. 111/60 Cena zł 20.—*

WARSZAWSKA DRUKARNIA NAUKOWA, W-WA, ŚNIADECKICH 8

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VIII, No. 4, 1960

MATHEMATICS

On the Uniqueness of the Limit Problem for the Ultrahyperbolic Equation

by

J. RADZIKOWSKI

Presented by T. WAŻEWSKI on January 12, 1960

In a Riemannian manifold with ultrahyperbolic signature we consider a domain Ω_{k+n} with compact closure and in that domain an ultrahyperbolic equation of the form

$$\Delta_2 u \equiv \sum_{i=1}^{k} g_{ii}(x) \frac{\partial^2 u}{\partial x_i^2} - \sum_{j=k+1}^{k+n} g_{jj}(x) \frac{\partial^2 u}{\partial x_j^2} = 0 , \tag{1}$$

where the metric form coefficients $g_{ii}(x) \in C^1(\overline{\Omega}_{k+n})$ for $i = 1, 2, ..., k+n$.

The aim of this paper is to give a uniqueness theorem for the solution of Eq. (1) in a domain σ_{k+n}, which will be more fully discussed below, belonging to the domain Ω_{k+n}. This paper is a modification of a method of L. Hörmander concerning the question of uniqueness of limit problems for normally hyperbolic equations.

We shall now introduce several notions, which will be used later in this paper.

DEFINITION 1. An ultrahyperbolic space is a finite-dimensional vector space in which a symmetric bi-linear form $\langle \vec{\xi}, \vec{\eta} \rangle$ — which will be called a scalar product — is defined, such that

$$\langle \vec{\xi}, \vec{\xi} \rangle = [(\xi^1)^2 + (\xi^2)^2 + \cdots + (\xi^k)^2] - [(\xi^{k+1})^2 + (\xi^{k+2})^2 + \cdots + (\xi^{k+n})^2].$$

DEFINITION 2. $\|\vec{\xi}\|^2 = \langle \vec{\xi}, \vec{\xi} \rangle$.

I. Classification of vectors. Vector $\vec{\xi} \neq \vec{0}$ has a

1° temporal character, if $\|\vec{\xi}\|^2 > 0$,

2° isotropic character, if $\|\vec{\xi}\|^2 = 0$,

3° spatial character, if $\|\vec{\xi}\|^2 < 0$.

II. Classification of hyperplanes. A hyperplane is said to have a 1° spatial character, if the vector normal to it has a temporal character $\|\vec{n}\|^2 > 0$, 2° isotropic character if the vector normal to it has an isotropic character $\|\vec{n}\|^2 = 0$, 3° temporal character if the vector normal to it has a spatial character $\|\vec{n}\|^2 < 0$.

III. The character of a hypersurface at a given point is defined by the character of a hyperplane tangent to it at that point.

DEFINITION 3. A vector field $\vec{b}(x) = [b^1(x), b^2(x), ..., b^k(x); b^{k+1}(x), b^{k+2}(x), ..., b^{k+n}(x)]$ defined in a compact domain $\overline{\Omega}_{k+n}$ is called a B-field, if it satisfies the following conditions

1° functions $b^i(x) \in C^0(\overline{\Omega}_{k+n})$ for $i = 1, 2, ..., k+n$,

2° $$\inf_{x \in \overline{\Omega}_{k+n}} b^1(x) > \sqrt{k+n-1} \max_{j=2,3,...,k+n} \sup_{x \in \overline{\Omega}_{k+n}} |b^j(x)|.$$

It is easy to prove the following property of B-fields.

1° B-fields form a convex cone, i.e. if ${}_1\vec{b}(x)$ and ${}_2\vec{b}(x)$ constitute B-fields, then $\vec{b}(x) = \alpha\, {}_1\vec{b}(x) + \beta\, {}_2\vec{b}(x)$, where $\alpha, \beta \geqslant 0$, is also a B-field.

2° A B-field has a temporal character, i.e. if $\vec{b}(x)$ is a B-field, then $\|\vec{b}(x)\|^2 > 0$.

3° If ${}_1\vec{b}(x)$ and ${}_2\vec{b}(x)$ are B-fields, then $\langle {}_1\vec{b}(x), {}_2\vec{b}(x) \rangle > 0$.

The main point of the proof of the theorem on the uniqueness of the limit problem for the ultrahyperbolic equation is the following lemma.

LEMMA. *Let $\vec{\xi}(x) \neq \vec{0}$ be an arbitrary, but fixed vector field belonging to class C° in domain Ω_{k+n} and let $\vec{a}(x)$ be the temporal field of class C° in that domain. Then for every point $\mathring{x} \in \Omega_{k+n}$, at which $\vec{\xi}(\mathring{x})$ is a temporal vector or a spatial vector, there exist a sphere $K(\mathring{x}, \varepsilon)$ and a Lorentz transformation $L = L(\mathring{x})$ such that*

$$2\langle \vec{a}(x), \vec{\xi}(x) \rangle \langle \vec{\xi}(x), L^{-1}\vec{b}(x) \rangle - \|\vec{\xi}(x)\|^2 \langle \vec{a}(x), L^{-1}\vec{b}(x) \rangle > 0$$

for every $x \in K(\mathring{x}, \varepsilon)$, where $\vec{b}(x)$ is a B-field in the domain Ω_{k+n}.

Proof. Consider the vector

$$\vec{c}(\mathring{x}) \overset{\text{def}}{=} 2\langle \vec{a}(\mathring{x}), \vec{\xi}(\mathring{x}) \rangle \vec{\xi}(\mathring{x}) - \|\vec{\xi}(\mathring{x})\|^2 \vec{a}(\mathring{x}),$$

where $\vec{\xi}(\mathring{x})$ is a temporal or a spatial vector. We shall show that this vector has the following properties:

1° $\vec{c}(\mathring{x}) \neq \vec{0}$,

2° $\vec{c}(\mathring{x})$ is a temporal vector.

1° Suppose that $\vec{c}(\mathring{x}) = \vec{0}$. Then $\vec{\xi}(\mathring{x}) = t\vec{a}(\mathring{x})$. Consequently

$$2t^2 \|\vec{a}(\mathring{x})\|^2 \vec{a}(\mathring{x}) - t^2 \|\vec{a}(\mathring{x})\|^2 \vec{a}(\mathring{x}) = \vec{0}.$$

Hence,

$$t^2 \|\vec{a}(\mathring{x})\|^2 \vec{a}(\mathring{x}) = \vec{0}.$$

By hypothesis $\vec{a}(\mathring{x})$ is a temporal vector, and thus $\|\vec{a}(x)\|^2 > 0$ and $\vec{a}(\mathring{x}) \neq \vec{0}$. Consequently, $t = 0$, i.e. $\vec{\xi}(\mathring{x}) = \vec{0}$, which contradicts the assumption. We have thus proved that $\vec{c}(\mathring{x}) \neq 0$.

$$2^\circ \quad \|\vec{c}(\mathring{x})\|^2 = 4[\langle \vec{a}(\mathring{x}), \vec{\xi}(\mathring{x})\rangle]^2 \|\vec{\xi}(\mathring{x})\|^2 - 4[\langle \vec{a}(x^2), \vec{\xi}(\mathring{x})\rangle]^2 \|\vec{\xi}(x^2)\|^2 + \\ + \|\vec{\xi}(\mathring{x})\|^4 \|\vec{a}(\mathring{x})\|^2 = \|\vec{\xi}(\mathring{x})\|^4 \|\vec{a}(\mathring{x})\|^2 > 0.$$

Consequently, $\vec{c}(\mathring{x})$ is a temporal vector.

Since $\vec{c}(\mathring{x})$ is a temporal vector, there exists a Lorentz transformation L such that the vector $L\vec{c}(\mathring{x})$ has the direction of versor $\vec{e}_1$, i.e. $L\vec{c}(\mathring{x}) = [c_1(\mathring{x}), 0, ..., 0; 0, ..., 0]$ where $c_1(\mathring{x}) > 0$. The continuity of $\vec{c}(x)$ implies the existence of a sphere $K(\mathring{x}, \varepsilon)$, in which $L\vec{c}(x)$ is a B-field. The properties of B-fields give us

$$\langle L\vec{c}(x), \vec{b}(x)\rangle > 0 \quad \text{for every} \quad x \in K(\mathring{x}, \varepsilon).$$

Thus, in the sphere $K(\mathring{x}, \varepsilon)$ we have

$$\langle \vec{c}(x), L^{-1}\vec{b}(x)\rangle > 0$$

and

$$2\langle \vec{a}(x), \vec{\xi}(x)\rangle \langle \vec{\xi}(x), L^{-1}\vec{b}(x)\rangle - \|\vec{\xi}(x)\|^2 \langle \vec{a}(x), L^{-1}\vec{b}(x)\rangle > 0.$$

Let us consider the domain $\bar{\sigma}_{k+n} \subset k(\mathring{x}, \varepsilon)$. Of course

$$2\langle \vec{a}(x), \vec{\xi}(x)\rangle \langle \vec{\xi}(x), L^{-1}b(x)\rangle - \|\vec{\xi}(x)\|^2 \langle \vec{a}(x), L^{-1}\vec{b}(x)\rangle > 0$$

for every $x \in \bar{\sigma}_{k+n}$.

Henceforth we shall concern ourselves with the domain $\bar{\sigma}_{k+n}$. .

The lemma proved above enables us to apply with suitable modification the argument introduced by Hörmander for the case of the normally hyperbolic equation.

Before we give the final results of the present paper, let us introduce several notions which appear in the final proposition.

DEFINITION 4. The field of a vector $\vec{\tilde{b}}(x)$ defined in a domain $\bar{\sigma}_{k+n}$ is called a $\tilde{B}$-field if

$$\vec{\tilde{b}}(x) \underset{x \in \bar{\sigma}_{k+n}}{\equiv} L^{-1}\vec{b}(x),$$

where $\vec{b}(x)$ is a B-field defined in the domain $\bar{\sigma}_{k+n}$, and L is a Lorentz transformation for the domain $\bar{\sigma}_{k+n}$.

DEFINITION 5. Every vector $\vec{\tilde{b}}(\mathring{x})$, where $\vec{\tilde{b}}(x)$ is a $\tilde{B}$-field is called a $\tilde{B}$-field.

DEFINITION 6. A hypersurface has at a given point the character $S^+_{\tilde{B}}$, if the vector normal to a hyperplane tangent to the hypersurface at that point is a $\tilde{B}$-vector.

DEFINITION 7. A hypersurface has at a given point the character $S^-_{\tilde{B}}$, if the vector normal to a hyperplane tangent to the given hypersurface at that point is a vector opposite to the $\tilde{B}$-vector.

DEFINITION 8. A hypersurface has at a given point the character S_0, if it has spatial character at that point but has neither the character $S^{+}_{\widetilde{B}}$ nor the character $S^{-}_{\widetilde{B}}$ at that point.

THEOREM. *If $u \in C^2(D_{k+n})$ and $u \in C^1(\overline{D}_{k+n})$ is a solution of the ultrahyperbolic equation $\Delta_2 u = 0$ in a sufficiently small domain D_{k+n} (i.e. $\overline{D}_{k+n} \subset K(\mathring{x}, \varepsilon)$, where $K(\mathring{x}, \varepsilon)$ is the sphere mentioned in the Lemma) satisfying the assumptions:*

1° *$\overline{D}_{k+n}$ is a compact domain,*

2° *the boundary of a domain D_{k+n} is regular,*

3° *there exists a function $\varphi(x) \in C^1(\overline{D}_{k+n})$ such that $\overrightarrow{grad}\ \varphi(x)$ has a temporal character, and the boundary of domain D_{k+m} consists of the surfaces $S^{+}_{\widetilde{B}}$, $S^{-}_{\widetilde{B}}$, S_0, S_{ch} and S_t, and function u satisfies the limit conditions:*

$$1^\circ \quad u = \frac{du}{dn} = 0 \quad \text{on} \quad S^{+}_{\widetilde{B}},$$

$$2^\circ \quad u = 0 \quad \text{on} \quad S^{-}_{\widetilde{B}},$$

$$3^\circ \quad u = \frac{du}{dn} = 0 \quad \text{on} \quad S_0,$$

$$4^\circ \quad u = 0 \quad \text{on} \quad S_{ch},$$

$$5^\circ \quad u = 0 \quad \text{on} \quad S_t,$$

then $u \equiv 0$ in that domain.

Owens gives in his paper a theorem on the uniqueness of the solution of the ultrahyperbolic equation of the form

$$L[u] \equiv \sum_{i=1}^{k} \frac{\partial^2 u}{\partial x_i^2} - \sum_{j=k+1}^{k+n} \frac{\partial^2 u}{\partial x_j^2} = 0 \tag{2}$$

in a unit sphere K.

Let us make the transformation

$$x_i' = \delta x_i, \qquad \delta > 0, \qquad i = 1, 2, \ldots, k+n.$$

Then the above equation will change into an equation of the same form, namely into the equation

$$\sum_{i=1}^{k} \frac{\partial^2 u}{\partial x_i'^2} - \sum_{j=k+1}^{k+n} \frac{\partial^2 u}{\partial x_j'^2} = 0,$$

and the unit sphere K will change into a sphere K_δ with radius δ. If δ is sufficiently small, then $K_\delta \subset D_{k+n}$.

Let us now compare the assumptions of Owens with those of our Theorem.

Owens assumes that, on the spatial and the characteristic part of the boundary of the unit sphere K, the solution of the ultrahyperbolic equation [2] satisfies the limit conditions $u = 0$ and $du/dn = 0$, and on the temporal part — the limit condition $u = 0$. Comparing the above assumptions with those of our Theorem we can see that Owens' assupmtions are stronger. The results obtained in this paper are thus a generalization of Owens' results.

The author wishes to thank Docent K. Maurin for his kind help and advice in connection with this work.

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https://archive.org/details/phytopathologia_1960_8_4

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VIII, No. 4, 1960

MATHEMATICS

On Two Elliptic Systems of Second Order Linear Differential Equations

by

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Presented by S. MAZUR on January 30, 1960

In this note we consider two examples of equation systems for which problems of the Dirichlet type behave in different ways, namely A. W. Bicadze's system [1] and the system of two Laplace operators. It appears that the boundary problems for those systems with an oblique derivative in the boundary condition have similar properties.

1° Consider the following boundary problems for the system of A. W. Bicadze

$$\left[\frac{\partial^2 u(x)}{\partial x_1^2} - \frac{\partial^2 u(x)}{\partial x_2^2}\right] + 2\mathcal{C}\frac{\partial^2 u(x)}{\partial x_1 \partial x_2} = 0, \qquad x = (x_1, x_2) \in V \tag{1}$$

$$\lim_{x \to y} \sum_{i=1,2} \left[\{E[\nu_1^2(y) - \nu_2^2(y)] + 2\mathcal{C}\nu_i(y)\nu_2(y)\}\nu_i(y) + B(y)\tau_i^0(y)\right]\frac{\partial u(x)}{\partial x_i} + C(y)u(y) = 0, \qquad y = (y_1, y_2) \in S, \tag{2}$$

where

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix},$$

the domain V is convex, bounded with a curve S triply smooth; $\nu(y) = (\nu_1(y), \nu_2(y))$ is a unit vector of the interior normal to S at the point y, and $\tau^0(y) = (\tau_1^0(y), \tau_2^0(y))$ is a unit vector of the tangent. The matrices

$$B(y) = \begin{pmatrix} b_{11}(y) & b_{12}(y) \\ b_{21}(y) & b_{22}(y) \end{pmatrix}, \qquad C(y) = \begin{pmatrix} c_{11}(y) & c_{12}(y) \\ c_{21}(y) & c_{22}(y) \end{pmatrix}$$

are differentiable in a continuous manner on S, the first of them twice.

The left-hand side of (2) has the general form of a regular boundary condition for system (1), [2].

Problems (1), (2) are considered in the class of columns $u(x)$ continuous in $V \cup S$, twice differentiable in a continuous manner in V, for which there exists a uniform limit on the left-hand side of (2).

J. B. Łopatyński's condition of reducibility of the problems (1), (2) to a system of regular integral equations, i.e. the condition of a finite number

of linearly independent own solutions of that problem, can be expressed as follows: for any point $y \in S$ and an arbitrary non-zero real vector $\tau(y) = (\tau_1(y), \tau_2(y))$, $(\nu(y), \tau(y)) = 0$ on S [3],

$$(3)\qquad \operatorname{rank}\operatorname*{res}_{\lambda = i\,|\tau(y)|}\left\{\frac{1}{(\lambda^2 + |\tau(y)|^2)^2}\times\right.$$
$$\times\left[\{E[\nu_1^2(y) - \nu_2^2(y)] + 2\mathcal{C}\,\nu_1(y)\,\nu_2(y)\}\,\lambda + B(y)\,(\tau^0(y), \tau(y))\right]\times$$
$$\times\left[E\{[\nu_1(y)\,\lambda + \tau_1(y)]^2 - [\nu_2(y)\,\lambda + \tau_2(y)]^2\} -\right.$$
$$\left.\left. - 2\mathcal{C}[\nu_1(y)\,\lambda + \tau_1(y)]\,[\nu_2(y)\,\lambda + \tau_2(y)]]\,(E, E\lambda)\right\} = 2\,.$$

On performing the multiplication under the integral sign and also the remaining operations, we can write down the condition in the following form:

$$\operatorname{rank}\,(iE \pm \mathcal{C}, i|\tau(y)|\,[2E \mp B(y)\,\{[\nu_1^2(y) - \nu_2^2(y)]\,i \pm 2\nu_1(y)\,\nu_2(y)\}]\,(iE \pm \mathcal{C})) = 2.$$

On account of $\det(iE \pm \mathcal{C}) = 0$, this condition is equivalent to the condition that an arbitrary determinant, whose first column is one of the first two columns of the matrix on the left side and whose second column is one of the remaining columns of that matrix should not be equal to zero, for instance (the factor $i\,|\tau(y)| \neq 0$ being omitted):

$$\begin{vmatrix} \mp 1 & i[2 \mp b_{11}(y)]\,\{[\nu_1^2(y) - \nu_2^2(y)]\,i \pm 2\nu_1(y)\,\nu_2(y)\} - \\ & \quad - b_{12}(y)\,\{[\nu_1^2(y) - \nu_2^2(y)]\,i \pm 2\nu_1(y)\,\nu_2(y)\} \\ i & \mp i b_{21}(y)\,\{[\nu_1^2(y) - \nu_2^2(y)]\,i \pm 2\nu_1(y)\,\nu_2(y)\} \pm 2 - \\ & \quad - b_{22}(y)\,\{[\nu_1^2(y) - \nu_2^2(y)]\,i \pm 2\nu_1(y)\,\nu_2(y)\} \end{vmatrix} =$$

$$= \{\pm[b_{22}(y) - b_{11}(y)] + i[b_{12}(y) + b_{21}(y)]\}\,\{[\nu_1^2(y) - \nu_2^2(y)]\,i \pm 2\nu_1(y)\,\nu_2(y)\} \neq 0\,,$$

i. e.

$$(4)\qquad b_{22}(y) - b_{11}(y) + i[b_{12}(y) + b_{21}(y)] \neq 0 \qquad \text{on } S\,.$$

Hence, it follows that the condition of reducibility of boundary conditions to regular integral equations for problems with an oblique derivative in the boundary condition cannot be at all simplified in the same manner as in the case of problems of the Dirichlet type [4].

Condition (4) is at the same time an analogous condition for the conjugate problem, [2]:

$$(1^*)\qquad \left|\frac{\partial^2 v(x)}{\partial x_1^2} - \frac{\partial^2 v(x)}{\partial x_2^2}\right| - 2\mathcal{C}\,\frac{\partial^2 v(x)}{\partial x_1\,\partial x_2} = 0\,, \qquad (x \in V)$$

$$(2^*)\qquad \lim_{x \to y} \sum_{i=1,2} |\{E[\nu_1^2(y) - \nu_2^2(y)] - 2\mathcal{C}\,\nu_1(y)\,\nu_2(y)\}\,\nu_i(y) +$$
$$+ \{4E\,\nu_1(y)\,\nu_2(y) - 2\mathcal{C}\,[\nu_1^2(y) - \nu_2^2(y)] - B^*(y)\}\,\tau_i^0(y)|\,\frac{\partial v(x)}{\partial x_i} +$$
$$+\left[\sum_{i=1,2} \tau_i^0(y)\,\frac{\partial\{2E\nu_1(y)\,\nu_2(y) + \mathcal{C}[\nu_1^2(y) - \nu_2^2(y)] - B^*(y)\}}{\partial y_i} + C^*(y)\right] v(y) = 0 \quad (y \in S)$$

($\tau^0(y) = (\nu_2(y), -\nu_1(y))$, A^* denoting the transposed matrix A).

Indeed, the space of solutions of problems (1*), (2*) can be mapped in a bi-unique manner upon the space of solutions of the problem for system (1) with the following boundary condition:

$$(\tilde{2}) \qquad \lim_{x\to y} \sum_{i=1,2} [\{E[\nu_1^2(y)-\nu_2^2(y)]+2\mathcal{E}\nu_1(y)\nu_2(y)\}\nu_i(y)+$$

$$+\{4E\nu_1(y)\nu_2(y)-2\mathcal{E}[\nu_1^2(y)-\nu_2^2(y)]-\mathcal{F}B^*(y)\mathcal{F}\}\tau_i^0(y)]\frac{\partial u(x)}{\partial x_i}+$$

$$+\left[\sum_{i=1,2}\tau_i^0(y)\frac{\partial\{2E\nu_1(y)\nu_2(y)-\mathcal{E}[\nu_1^2(y)-\nu_2^2(y)]-\mathcal{F}B^*(y)\}}{\partial y_i}+\mathcal{F}C^*(y)\mathcal{F}\right]\times$$

where $\qquad \times u(y)=0, \qquad (y\in S),$

$$\mathcal{F}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \mathcal{F}B^*(y)\mathcal{F}=\begin{pmatrix}b_{22}(y) & b_{12}(y)\\ b_{21}(y) & b_{11}(y)\end{pmatrix}$$

according to the formulae $v(x)=\mathcal{F}u(x)$, $u(x)=\mathcal{F}v(x)$.

(We can also write directly the condition of reducibility of problem (1*), (2*) to integral equations and then reduce it to the analogical form (3) by multiplying each of the first two factors-matrices by $\mathcal{F}$).

Each boundary problem may be assigned to a curve ($b_{11}(y)$, $b_{12}(y)$, $b_{21}(y)$, $b_{22}(y)$) in the four-dimensional space (b_{11}, b_{12}, b_{21}, b_{22}), in which the manifold:

$$(5) \qquad b_{11}=b_{22}, \qquad b_{12}=-b_{21}$$

is a two-dimensional plane.

The set of all such curves for which for any $y\in S$

$$b_{22}(y)-b_{11}(y)+i[b_{12}(y)+b_{21}(y)]\neq 0$$

has a countable quantity of connection components which can be distinguished e.g. by such representatives of their elements as

$$(-\cos m\varphi, 0, \sin m\varphi, 0) \qquad (m=0,\pm 1,\pm 2,\ldots),$$

where

$$y_1=\varrho\cos\varphi, \qquad y_2=\varrho\sin\varphi, \qquad (y_1,y_2)=y\in S, \qquad 0\leqslant\varphi<2\pi, \qquad 0\in V.$$

From the independence of the index of problems (1), (2), of the coefficient $C(y)$, [5], and from a theorem in [6] it follows that in each component the index of problems (1), (2) is the same.

Let us first consider the set of problems for which the suitable curves have common points with the plane (5). The general solution of the system (1) in the unit circle (see e.g. [7]) can be written in the form:

$$(6) \qquad u(r,\varphi)=r(E\cos\varphi-\mathcal{E}\sin\varphi)A_0+$$

$$+\sum_{n=0}^{\infty} r^n[(r^2A_{n+1}+B_n)\cos n\varphi+\mathcal{E}(r^2A_{n+1}+B_n)\sin n\varphi]$$

$$(0\leqslant r\leqslant 1, \qquad 0\leqslant\varphi<2\pi),$$

where A_i and B_i are constant columns of height 2. Substituting this expression into boundary condition (2), where $B(y) = \text{const.}$, $C(y) \equiv 0$, we obtain

$$(7)\quad [-(B\mathcal{C}+E)A_0+B\mathcal{C}(A_2+B_1)]\cos\varphi-[(B+\mathcal{C})A_0+B(A_2+B_1)]\sin\varphi+$$
$$+\sum_{n=2}^{\infty}\{[-(nA_{n-1}+(n-2)B_{n-2})+nB\mathcal{C}(A_{n+1}+B_n)]\cos n\varphi-$$
$$-[\mathcal{C}(nA_{n-1}+(n-2)B_{n-2})+nB(A_{n+1}+B_n)]\sin n\varphi\}=0.$$

Assuming first that $B=0$, we obtain

$$A_0=0,\qquad A_{n+1}=-\frac{n}{n+2}B_n,$$

whence

$$u_n(r,\varphi)=r^n\left[E\left(-\frac{n}{n+2}r^2+1\right)\cos n\varphi+\mathcal{C}\left(-\frac{n}{n+2}r^2+1\right)\sin n\varphi\right]B_n$$
$$(n=0,1,\ldots)$$

is an infinite system of linearly independent solutions of the problem. Generally, if $B=Ea+\mathcal{C}b$, where $a^2+b^2\neq 0$, then

$$A_0=0,\ B_1=-A_2,\ B_{2k}=\frac{1}{k}\sum_{i=1}^{k}\frac{(-1)^i}{(a^2+b^2)^i}(\mathcal{C}a+Eb)^i A_{2(k-i)+1}-A_{2k+1},\ B_{2k+1}=$$
$$=\frac{2}{2k+1}\sum_{i=1}^{k}\frac{(-1)^i}{(a^2+b^2)^i}(\mathcal{C}a+Eb)^i A_{2(k-i)+2}-A_{2(k+1)}\qquad(k=1,2,\ldots)$$

and

$$u_0(r,\varphi)=r(E\cos\varphi-\mathcal{C})A_0,\ u_{2k}(r,q)=r^{2k-1}[E(r^2-1)\cos(2k-1)\varphi+\mathcal{C}(r^2-1)\times$$
$$\times\sin(2k-1)\varphi]A_{2k}+2\sum_{n=1}^{\infty}\frac{(-1)^n r^{2(k+2)-1}}{|2(k+n)-1|(a^2+b^2)^n}\times$$
$$\times\{E\cos[2(k+n)-1]\varphi+\mathcal{C}\sin[2(k+n)-1]\varphi\}(\mathcal{C}a+Eb)^n A_{2k},$$
$$u_1(r,\varphi)=A_1,\ u_{2k+1}(r,\varphi)=r^{2k}[E(r^2-1)\cos 2k\varphi+\mathcal{C}(r^2-1)\sin 2k\varphi]A_{2k+1}+$$
$$+\sum_{n=1}^{\infty}\frac{(-1)^n r^{2(k+n)}}{(k+n)(a^2+b^2)^n}[E\cos 2(k+n)\varphi+\mathcal{C}\sin 2(k+n)\varphi]\times$$
$$\times(\mathcal{C}a+Eb)^n A_{2k+1}\qquad(k=1,2,\ldots)$$

will be the full system of linearly independent solutions. Here A, just as B in the preceding formulae, are entirely arbitrary constant columns.

Using formula (6) and problems (1), $(\tilde{2})$ we can find in an analogous way that the conjugate problems also have an infinite number of linearly independent solutions. However, the set of these problems contains also regular problems.

Indeed, with

$$B=0\ \text{ and }\ C=\begin{pmatrix}C_{11}&C_{12}\\C_{12}&-C_{11}\end{pmatrix}=\text{const.},\ (\mathcal{C}C\mathcal{C}=C)\ C_{11}^2+C_{12}^2>0,\ \neq 1;$$

it follows from (6) and (7) that the unique solution of this problem is the zero solution. As before, we can verify that the conjugate problem for that problem will have the same property.

Similarly, using (6) it can be proved that the index of problems (1), (2) in the connection component with the representative $(-\cos m\varphi,\ 0,\ \sin m\varphi,\ 0)$ is equal to $2m$.

In the case $m = 0$ the unique solution of the problem in a unit circle for $B(\varphi) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ is an arbitrary constant column, A, and for the corresponding problem (1), $(\widetilde{2})$ $-r^2 A$.

In the case $m > 0$, the unique solution of (1), (2) for $B(\varphi) = \begin{pmatrix} -\cos m\varphi & 0 \\ \sin m\varphi & 0 \end{pmatrix}$ remain an arbitrary constant column, and the full system of linearly independent solutions of the corresponding problems (1), $(\widetilde{2})$ can be written in the following form:

$$u_1(r,\varphi) = \left\{ r^2 + \frac{r^2 m}{2} \left[((m+1)r^2 - m) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cos 2m\varphi + \right.\right.$$

$$\left.\left. + ((m+1)r^2 - m)\, \mathcal{F} \sin 2m\varphi \right] \right\} A_1,$$

$$u_{i+1}(r,\varphi) = \left\{ r^2 \left[\left(r^2 - \frac{i}{i+2} \right) E \cos i\varphi + \left(r^2 - \frac{i}{i+2} \right) \mathcal{C} \sin i\varphi \right] + \right.$$

$$+ \frac{r^{2m-i}}{i+2} \left[((2m+2-i)r^2 - (2m-i)) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cos (2m-i)\varphi - \right.$$

$$\left.\left. - ((2m+2-i)r^2 - (2m-i))\, \mathcal{F} \sin (2m-i)\varphi \right] \right\} A_{i+1} \qquad (i = 1, 2, \ldots, m-2)$$

$$u_m(r,\varphi) = \left\{ r^{m-1} \left[\left(r^2 - \frac{m-1}{m+1} \right) E \cos (m-1)\varphi + \right.\right.$$

$$+ \left(r^2 - \frac{m-1}{m+1} \right) \mathcal{C} \sin (m-1)\varphi \left] + r^{m+1} \left[\left(\frac{m+3}{m+1} r^2 - 1 \right) \begin{pmatrix} 1 & 0 \\ 0 & 1/m \end{pmatrix} \cos (m+1)\varphi + \right.\right.$$

$$\left.\left. + \left(\frac{m+3}{m+1} r^2 - 1 \right) \begin{pmatrix} 0 & -1/m \\ 1 & 0 \end{pmatrix} \sin (m+1)\varphi \right] \right\} A_m$$

$$u_{m+1}(r,\varphi) = r^m \left[\left(r^2 - \frac{m}{m+2} \right) E \cos m\varphi + \left(r^2 - \frac{m}{m+2} \right) \mathcal{C} \sin m\varphi \right] A_{m+1},$$

where $A_1, \ldots, A_{m+1}$ are arbitrary constant columns.

2° Let us now consider a boundary problem with an oblique derivative for a system of two Laplace equations

$$\Delta u(x) = 0, \qquad x = (x_1, \ldots, x_n) \in V \tag{8}$$

$$\lim_{x \to y} \sum_{i=1}^{n} [\nu_i(y) + B_i(y)] \frac{\partial u(x)}{\partial x_i} = 0 \qquad y = (y_1, \ldots, y_n) \in S. \tag{9}$$

The domain V, surface S and matrices $B_i(y)$ have the same properties, respectively, as in the preceding example. Moreover, $\sum_{i=1}^{n} B_i(y)\nu_i(y) = 0$ ($\nu(y) = (\nu_1(y), ..., \nu_n(y))$ is a unit vector of the interior normal to S). Problems (8), (9) are considered in a class with the same properties as those possessed by the class in which we have considered problems (1), (2). Condition (9) has the general form of a regular boundary condition with an oblique derivative for system (8), [2].

This problem is reducible to a system of integral equations, [3], if for every point $y \in S$ and an arbitrary non-zero real vector $\tau(y) = (\tau_1(y), ..., \tau_n(y))$ $(\tau_n(y)), (\nu(y), \tau(y)) = 0$ on S we have the

$$\operatorname*{row\,res}_{\lambda = i|\tau(y)|} \left\{ \frac{1}{\lambda^2 + |\tau(y)|^2} \left[E\lambda + \sum_{i=2}^{n} B_i(y)\,\tau_i(y) \right] (E, E\lambda) \right\} = 2 .$$

That is, for any $y \in S$ and an arbitrary real unit vector $\tau(y)$ tangent to S in this point, the own values of the matrix

$$\sum_{i-1}^{n} B_i(y)\,\tau_i(y) \tag{10}$$

cannot assume the value i.

This condition is also the condition of reducibility to regular integral equations of the problem conjugate with problems (8), (9) because the boundary condition of that problem according to formulae [2] will be written as follows:

$$\lim_{x \to y} \sum_{i=1}^{n} [\nu_i(y) - B_i^*(y)] \frac{\partial v(x)}{\partial x_i} + \left\{ \sum_{i,j-1}^{n} \left[\frac{\partial [B_j^*(y)\,\nu_i(y) - B_i^*\,\nu_j(y)]}{\partial y_i} \nu_j(y) - \right. \right.$$

$$\left. \left. - \frac{\partial [B_j^*(y)\,\nu_i(y) - B_i^*(y)\,\nu_j(y)]}{\partial y_j} \right] \nu_i(y) \right\} v(y) = 0 \qquad (y \in S) . \tag{9*}$$

In the case $n = 2$, the condition of reducibility to a system of regular integral equations comes to that for any $y \in S$

$$-1 + b_{11}(y)\,b_{22}(y) - b_{12}(y)\,b_{21}(y) + i[b_{11}(y) + b_{22}(y)] \neq 0 .$$

Just as for 1° the set of all curves $(b_{11}(y), b_{12}(y), b_{21}(y), b_{22}(y))$, for which this condition is void has a countable quantity of connection components which can be distinguished e.g. by such representatives of the elements

$$(\sin m\varphi, -1, 1 + \cos m\varphi, 0) \qquad (m = 0, \pm 1, \pm 2, ...) .$$

From a theorem in [6], and from the representation of harmonic function in the unit circle by the series of circle functions, (9) and (9*) follows, that in each component the index is the same and equal to $2m$.

Actually, for $m = 0$ the unique solution of the problem (8), (9) in the unit circle for $B(y) = \text{const.}$, $\det(Ei + B) \neq 0$ is an arbitrary constant column, similarly as the problem (8), (9*).

For $m > 0$ and $B(\varphi) = \begin{pmatrix} \sin m\varphi & -1 \\ 1 + \cos m\varphi & 0 \end{pmatrix}$ full system of the linearly independent solutions of the problem (8), (9) in the unit circle can be written in the form: for $m = 2k+1$

$$u_0(r, \varphi) = A_0$$

$$u_i(r, \varphi) = \left\{ r^i E \cos i\varphi + \frac{2i}{m-i} r^{m-i} \left[E \cos(m-i)\varphi + \right.\right.$$

$$\left. + \begin{pmatrix} 0 & -1 \\ 3/4 & 0 \end{pmatrix} \sin(m-i)\varphi \right] + \frac{i}{2(m+i)} r^{m+i} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \sin(m+i)\varphi +$$

$$\left. + \frac{i}{2m-i} r^{2m-i} \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cos(2m-i)\varphi + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \sin(2m-i)\varphi \right] \right\} A_i,$$

$$u_{k+i}(r, \varphi) = \left\{ r^i E \sin i\varphi + \frac{2i}{m-i} r^{m-i} \left[\begin{pmatrix} 0 & -1 \\ 3/4 & 0 \end{pmatrix} \cos(m-i)\varphi - E \sin(m-i)\varphi \right] - \right.$$

$$- \frac{i}{2(m+i)} r^{m+i} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cos(m+i)\varphi + \frac{i}{2m-i} r^{2m-i} \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cos(2m-i)\varphi - \right.$$

$$\left. \left. - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \sin(2m-i)\varphi \right] \right\} A_{k+i} \qquad (i = 1, \ldots, k).$$

In addition for $m = 2k+2$

$$u_{2k+1}(r, \varphi) = \left\{ r^{k+1} \left[E \cos(k+1)\varphi + \begin{pmatrix} 0 & -2,3 \\ 1/2 & 0 \end{pmatrix} \sin(k+1)\varphi + \right.\right.$$

$$\left. \left. + r^{3k+3} \left[\begin{pmatrix} 0 & 0 \\ 0 & 1,9 \end{pmatrix} \cos(3k+3)\varphi + \begin{pmatrix} 0 & 0 \\ 1,6 & 0 \end{pmatrix} \sin(3k+3)\varphi \right] \right\} A_{2k+1}.$$

The corresponding problem (8), (9*) has only zero solution.

If $B(y) = \text{const.}$, and $\det Ei + B) = 0$ the problem (8), (9) has an infinite quantity of linearly independent solutions of the form

$$u_0(r, \varphi) = A_0, \qquad u_i(r, \varphi) = (E \cos i\varphi + B \sin i\varphi) r^i A_i \qquad (i = 1, 2, \ldots).$$

In an arbitrary space $n \geqslant 3$ all the problems (8), (9) can be assigned to the space of matrix $(B_1(y), \ldots, B_n(y))$ determined and twice differentiable in a continuous manner on S, for which

$$\sum_{i=1}^{n} B_i(y) \nu_i(y) = 0 \qquad (y \in S)$$

and it has been proved, that this space divides into two connected parts, one of which is starry towards the zero matrix and fulfills the condition:

for any $y \in S$

$$-(b_{11}(y), \tau(y))(b_{22}(y), \tau(y)) + (b_{12}(y), \tau(y))(b_{21}(y), \tau(y)) + 1 > 0,$$

where

$$b_{ij}(y) = (b_{ij}^1(y), ..., b_{ij}^n(y)) \ (i, j = 1, 2), \ B_k(y) = \begin{pmatrix} b_{11}^k(y) & b_{12}^k(y) \\ b_{21}^k(y) & b_{22}^k(y) \end{pmatrix} \quad (k = 1, ..., n)$$

and $\tau(y)$ is the arbitrary unit vector tangent to S in the point y, perpendicular to the vector $b_{11}(y) + b_{22}(y)$, if in this point it differs from zero, and arbitrary at all, when $b_{11}(y) + b_{22}(y) = 0$. This latter part has such a property, that for each appertaining matrix $(B_1(y), ..., B_n(y))$ the condition (10) is fulfilled.

For each matrix appertaining to the second part of that space such a point $y \in S$ exists in which

$$-(b_{11}(y), \tau(y))(b_{22}(y), \tau(y)) + (b_{12}(y), \tau(y))(b_{21}(y), \tau(y)) + 1 = 0$$

for a certain unit vector $\tau(y)$ tangent to S in this point. In this point the own value of the matrix assumes the value of i for at least two and for at most four values of vector $\tau(y)$.

Hence, and from the theorem in [6] follows the

THEOREM. *Under the given conditions each boundary problem* (8), (9). *reducible after J. B. Łopatyński to a system of regular integral equation, has an index zero.*

In the case of the Dirichlet type problem, for an elliptic system of two linear equations in a space, such a theorem has been proved by B. Bojarski (see [8]).

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VIII, No. 4, 1960

MATHEMATICS

On the Isomorphisms of the Spaces H_α and m

by

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Presented by W. ORLICZ on February 6, 1960

Throughout this note we shall use the following notations:

$\langle C, \|\ \|_0 \rangle$ — the space of all real functions $x(t)$ continuous in $\langle 0,1\rangle$ and vanishing at zero, with the norm $\|x\|_0 = \max\limits_{\langle 0,1\rangle} |x(t)|$,

$\langle m, \|\ \| \rangle$ — the space of all bounded real sequences $\xi = (\xi_1, \xi_2, \ldots)$ with the norm $\|\xi\|$ defined as $\sup\limits_n |\xi_n|$,

$\langle c_0, \|\ \| \rangle$ — the subspace of $\langle m, \|\ \| \rangle$ of all sequences $\xi = (\xi_1, \xi_2, \ldots)$ such that $\lim\limits_n \xi_n = 0$,

$\langle l, \|\ \|_l \rangle$ — the space of all real sequences $a = (a_1, a_2, \ldots)$ such that $\|a\|_l = \sum\limits_n |a_n| < \infty$,

$\langle H_\alpha, \|\ \|_\alpha \rangle$ — the space of all functions $x \in C$ for which

$$|x|_\alpha = \sup_{t_1, t_2 \in \langle 0,1\rangle} \frac{|x(t_1) - x(t_2)|}{|t_1 - t_2|^\alpha} < \infty, \tag{1}$$

$\langle H_{\alpha,0}, \|\ \|_\alpha \rangle$ — the subspace of $\langle H_\alpha, \|\ \|_\alpha \rangle$ of all functions $x \in C$ satisfying the following condition

$$\lim_{\delta \to 0+} \sup_{\substack{|t_1 - t_2| \leq \delta \\ t_1, t_2 \in \langle 0,1\rangle}} \frac{|x(t_1) - x(t_2)|}{|t_1 - t_2|^\alpha} = 0. \tag{2}$$

It is well known that the spaces $\langle C, \|\ \|_0 \rangle$, $\langle m, \|\ \| \rangle$; $\langle c_0, \|\ \| \rangle$, $\langle l, \|\ \|_l \rangle$ and $\langle H_\alpha, \|\ \|_\alpha \rangle$, $0 < \alpha \leq 1$, are Banach spaces, and, as has been proved by J. Musielak and Z. Semadeni [5]

The space $\langle H_{\alpha,0}, \|\ \|_\alpha \rangle$ $(0 < \alpha < 1)$ *is a separable Banach space and*

$$H_{\alpha,0} = \overline{\bigcup_{\varepsilon > 0} H_{\alpha+\varepsilon}},$$

where the closure is taken in the sense of the norm $\|\ \|_\alpha$.

This result suggests the method of proof of Theorem 2.

We shall say two Banach spaces $\langle X_1, \|\ \|_1\rangle$ and $\langle X_2, \|\ \|_2\rangle$ are *isomorphic*, whenever there is a one-to-one linear operation from X_1 onto X_2. Now, let $\{\chi_n\}$, $n = 1, 2, \ldots$, denote the orthonormal set of Haar functions defined as follows

$$\chi_1(t) \equiv 1,\quad \chi_{2^n+k}(t) = \begin{cases} \sqrt{2^n} & \text{for } \left\langle \frac{2k-2}{2^{n+1}}, \frac{2k-1}{2^{n+1}} \right), \\ -\sqrt{2^n} & \text{for } \left(\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}} \right\rangle, \\ 0 & \text{elsewhere in } \langle 0, 1\rangle, \end{cases} \tag{3}$$

for $n = 0, 1, \ldots;\ k = 1, \ldots, 2^n$. Let

$$\varphi_n(t) = \int_0^t \chi_n(\tau)\, d\tau \quad \text{for} \quad t \in \langle 0, 1\rangle,\ n = 1, 2, \ldots. \tag{4}$$

Moreover, let

$$\chi_1^{(\alpha)} = \chi_1,\ \varphi_1^{(\alpha)} = \varphi_1 \quad \text{and} \quad \chi_{2^n+k}^{(\alpha)} = \frac{2^{(n+1)\alpha}}{2\sqrt{2^n}}\, \chi_{2^n+k},\ \varphi_{2^n+k}^{(\alpha)} = \frac{2}{2^{(n+1)\alpha}}\, \varphi_{2^n+k} \tag{5}$$

for $n = 0, 1, \ldots;\ k = 1, \ldots, 2^n$. It is easy to verify that $\|\varphi_n^{(\alpha)}\|_\alpha = 1$ for $n = 1, 2, \ldots$. In the sequel the following lemma [2] will be applied:

LEMMA. *For every $x \in C$ the series*

$$\sum_{n=1}^{\infty} \left[\int_0^1 \chi_n^{(\alpha)}(\tau)\, dx(\tau) \right] \varphi_n^{(\alpha)}$$

converges to x with respect to the norm $\|\ \|_0$.

THEOREM 1. *Let $0 < \alpha < 1$. The spaces $\langle H_\alpha, |\ |_\alpha\rangle$ and $\langle m, |\ |\rangle$ are isomorphic. The isomorphism T_α from m onto H_α is given by*

$$x = \sum_{n=1}^{\infty} \xi_n\, \varphi_n^{(\alpha)}, \tag{6}$$

where the series coverges with respect to the norm $\|\ \|_0$, and T_α^{-1} (the isomorphism from H_α onto m) is given by

$$\xi_n = \int_0^1 \chi_n^{(\alpha)}(\tau)\, dx(\tau), \quad n = 1, 2, \ldots. \tag{7}$$

Moreover,

$$\frac{2}{3(2^\alpha - 1)(2^{1-\alpha} - 1)} \leqslant \|T_\alpha\| \leqslant \frac{2}{(2^\alpha - 1)(2^{1-\alpha} - 1)}, \quad \|T_\alpha^{-1}\| = 1. \tag{8}$$

Remark. The method of the proof of this Theorem cannot be applied in the case $\alpha = 1$; however, A. Pełczyński in [6] proved the isomorphism between $\langle H_1, \|\ \|_1\rangle$ and $\langle m, \|\ \|\rangle$.

Proof. The lemma shows that it is sufficient to prove (8). Since $|\xi| \leqslant |x|_\alpha$ for $\xi = T_\alpha^{-1} x$ and $x \in H_\alpha$, we have $\|T_\alpha^{-1}\| \leqslant 1$. Let $x(t) = t$ and $\xi = T_\alpha^{-1} x$. Then the equality $|\xi| = |x|_\alpha$ holds; hence $\|T_\alpha^{-1}\| = 1$. Now let us consider the function x defined by (6), where $\xi \in m$. We get

$$(9) \qquad x(t') - x(t'') \leqslant |\xi| \left[|t' - t''| + \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} |\varphi_{2^n+k}^{(\alpha)}(t') - \varphi_{2^n+k}^{(\alpha)}(t'')| \right]$$

for any $t', t'' \in \langle 0, 1 \rangle$, $t' \neq t''$. Let ν_0 denote a non-negative integer such that

$$(10) \qquad \frac{1}{2^{\nu_0 - 1}} < |t' - t''| \leqslant \frac{1}{2^{\nu_0}}.$$

It is quite easy to establish that

$$(11) \qquad \sum_{k=1}^{2^n} |\varphi_{2^n+k}^{(\alpha)}(t') - \varphi_{2^n+k}^{(\alpha)}(t'')| \leqslant \begin{cases} 2^{(1-\alpha)(n+1)} |t' - t''| & \text{for } 0 \leqslant n < \nu_0, \\ 2^{1-\alpha} 2^{-n\alpha} & \text{for } n \geqslant \nu_0. \end{cases}$$

The inequalities (9), (10) and (11) imply the following one

$$\frac{x(t') - x(t'')}{|t' - t''|^\alpha} \leqslant \frac{2}{(2^\alpha - 1)(2^{1-\alpha} - 1)} \|\xi\|;$$

hence,

$$\|T_\alpha\| \leqslant \frac{2}{(2^\alpha - 1)(2^{1-\alpha} - 1)}.$$

Now we shall prove the second inequality for $\|T_\alpha\|$. Let

$$(12) \qquad x_N = \sum_{n=1}^{2^{N+1}} \varphi_n^{(\alpha)}$$

and

$$(13) \qquad t_0 = \frac{4}{3} \frac{1}{2^{\nu_0 + 1}} = \sum_{k=\nu_0+1}^{\infty} \frac{\varepsilon_k}{2^k}, \qquad \varepsilon_{\nu_0+\nu} = \frac{(-1)^{\nu+1}+1}{2}, \quad \nu = 1, 2, \ldots.$$

The conditions on the non-negative integers ν_0 and $N > \nu_0$ will be imposed later. By (12) and (13) we get

$$(14) \qquad t_0 + \sum_{n=0}^{\nu_0 - 1} \sum_{k=1}^{2^n} \varphi_{2^n+k}^{(\alpha)}(t_0) = t_0 \sum_{n=0}^{\nu_0} 2^{(1-\alpha)n} = t_0^\alpha \frac{\left[\left(\frac{4}{3}\right)^{1-\alpha} - t_0^{1-\alpha}\right]}{2^{1-\alpha} - 1}.$$

$$(15) \qquad \sum_{k=1}^{2^{\nu_0}} \varphi_{2^{\nu_0}+k}^{(\alpha)}(t_0) = \varphi_{2^{\nu_0}+1}^{(\alpha)}(t_0) = \frac{1}{2} \left(\frac{4}{3}\right)^{1-\alpha} t_0^\alpha,$$

and if $t_n = \sum_{\nu_0+1}^{n} \varepsilon_k / 2^k$, then

$$(16)\qquad \sum_{n=\nu_0+1}^{N}\sum_{k=1}^{2^n}\varphi^{(\alpha)}_{2^n+k}(t_0)=\sum_{n=\nu_0+1}^{N}2^{(1-\alpha)(n+1)}\min\left(t-t_n,\,t_n+\frac{1}{2^n}-t\right)$$

$$=\sum_{n=\nu_0+1}^{N}\frac{1}{2^{(n+1)\alpha}}\left[1-|2^{n+1}(t-t_n)-1|\right]$$

$$=\frac{2}{3}\sum_{n=\nu_0+1}^{N}2^{-\alpha(n+1)}=2^{\alpha-1}\left(\frac{4}{3}\right)^{1-\alpha}\frac{1-2^{-\alpha(N-\nu_0)}}{2^{\alpha}-1}.$$

However, (14), (15) and (16) imply

$$x_N(t_0)\,t_0^{-\alpha}=\left(\frac{4}{3}\right)^{1-\alpha}\frac{3-2^{1-\alpha}}{2}\,\frac{1}{(2^{\alpha}-1)(2^{1-\alpha}-1)}-$$

$$-\left[\frac{t_0^{(1-\alpha)}}{2^{1-\alpha}-1}+\frac{1}{2^{N\alpha}}\left(\frac{2}{3}\right)^{\alpha}\frac{t_0^{-\alpha}}{2^{\alpha}-1}\right].$$

Let $\varepsilon>0$ and let ν_0 be such that

$$\frac{t_0^{(1-\alpha)}}{2^{1-\alpha}-1}<\varepsilon/2$$

and $N=N(\nu_0)$ be such that

$$\frac{1}{2^{N\alpha}}\left(\frac{2}{3}\right)^{\alpha}\frac{t_0^{-\alpha}}{2^{\alpha}-1}<\varepsilon/2.$$

Then

$$\|x_N\|_{\alpha}\geq\left(\frac{4}{3}\right)^{1-\alpha}\frac{3-2^{1-\alpha}}{2}\,\frac{1}{(2^{\alpha}-1)(2^{1-\alpha}-1)}-\varepsilon>\frac{2}{3}\,\frac{1}{(2^{\alpha}-1)(2^{1-\alpha}-1)}-\varepsilon.$$

Since for $\xi=T_{\alpha}^{-1}x_N$, $\|\xi\|=1$, we have finally

$$\|T_{\alpha}\|>\frac{2}{3(2^{\alpha}-1)(2^{1-\alpha}-1)}.$$

THEOREM 2. *Let $0<\alpha<1$. The spaces $\langle H_{\alpha,0},\|\,\|_{\alpha}\rangle$ and $\langle c_0,\|\,\|\rangle$ are isomorphic. The set $\{\varphi_n\}$, $n=1,2,\ldots$ constitutes the Schauder basis in the space $\langle H_{\alpha,0},\|\,\|_{\alpha}\rangle$. The isomorphisms $T_{\alpha,0}$ from C_0 onto $H_{\alpha,0}$ and $T_{\alpha,0}^{-1}$ from $H_{\alpha,0}$ onto c_0 are given by* (6) *and* (7), *respectively. Moreover,*

$$(17)\qquad \frac{2}{3(2^{\alpha}-1)(2^{1-\alpha}-1)}\leq\|T_{\alpha,0}\|\leq\frac{2}{(2^{\alpha}-1)(2^{1-\alpha}-1)}\quad\text{and}\quad\|T_{\alpha,0}^{-1}\|=1.$$

Proof. First of all we note that by Theorem 1

$$c_0=T_{\alpha,0}^{-1}\Big(\bigcup_{\varepsilon>0}H_{\alpha+\varepsilon}\Big),$$

where the closure is taken in the norm $\|\ \|_\alpha$. Thus, the spaces $\langle c_0, |\ |\rangle$ and $\langle \bigcup_{\varepsilon>0} H_{\alpha+\varepsilon}, \|\ \|_\alpha\rangle$ are isomorphic. If $x \in H_{\alpha,0}$, then by (7) $\lim_n \xi_n = 0$, where $\xi = T_{\alpha,0}^{-1}$; hence $\xi \in c_0$. This shows as follows:

$$\bigcup_{\varepsilon>0} H_{\alpha+\varepsilon} \subset H_{\alpha,0} \subset \overline{\bigcup_{\varepsilon>0} H_{\alpha+\varepsilon}}, \tag{18}$$

since it is easy to verify that $H_{\alpha,0}$ is closed in $\langle H_\alpha, \|\ \|_\alpha\rangle$, thus (18) implies

$$H_{\alpha,0} = \overline{\bigcup_{\varepsilon>0} H_{\alpha+\varepsilon}}.$$

Finally, the method of the proof of (8) shows that (17) is true.

THEOREM 3. *Every linear functional f on $\langle H_{\alpha,0}, \|\ \|_\alpha\rangle$ is of the form*

$$f(x) = \sum_{n=1}^{\infty} a_n \int_0^1 \chi_n^{(\alpha)}(t)\, dx(t),$$

where $a \in l$ and

$$\|f\|_\alpha \leqslant \|T_{\alpha,0}^{-1}\|\, \|a\|_l, \quad \|a\|_l \leqslant \|T_{\alpha,0}\|\, \|f\|_\alpha,$$

and the constants $\|T_{\alpha,0}\|$, $\|T_{\alpha,0}^{-1}\|$ are the best possible. Here $\|f\|_\alpha$ denotes the norm of the functional f.

For the proof it is sufficient to apply Theorem 2, the representation theorem of the linear functionals on $\langle c_0, \|\ \|\rangle$ and the well known Banach Theorem (see [1], p. 100, Theorem 3).

THEOREM 4. *Let $0 < \alpha < 1$, $\alpha + \beta = 1$, $y \in H_\beta$. Moreover, let y be a function of bounded variation over $\langle 0, 1\rangle$ with the variation $V(y)$. The linear functional*

$$f(x) = \int_0^1 y(t)\, dx(t)$$

on $\langle H_{\alpha,0}, \|\ \|_\alpha\rangle$ satisfies the following inequality

$$\|f\|_\alpha \leqslant \left| \int_0^1 y(t)\, dt \right| + A(\alpha)\, [V(y)\, \|y\|_\beta]^{1/2},$$

where $A(\alpha) = 1/\sqrt{2}\,(2^{\alpha/2} - 1)$.

The easy proof of the last theorem is based on the corollary from Theorem 3 in [2], on the Theorem 3 and on the method of the proof of the Theorem 3 in [4]. The detailed computations will be omitted.

The preceding results may be generalized, if we replace the function t^α by the function $\omega(t)$ satisfying suitable conditions [3].

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VIII, No. 4, 1960

MATHEMATICS

On Subdirect Sums of Simple Rings with Unity. I.

by

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Presented by A. MOSTOWSKI on February 18, 1960

In this paper we shall consider the rings semi-simple in the sense of the Brown - McCoy radical [2] (i.e. subdirect sums of simple rings with unity).

The set of all maximal modular ideals of a semi-simple ring is regarded as a subspace of Jacobson's structure space [3] and the usual topology is introduced. Different algebraic properties of a ring and its ideals are formulated in terms of closure operation. The notions of special and completely non-special rings are introduced. This classification is based on what is called the minimal set of a ring (par. 2). The special and completely non-special part are defined in an arbitrary semi-simple ring (par. 3).

1. Let $\mathfrak{r}$ be a semi-simple ring. Let us denote by M the set of all maximal modular ideals *) of $\mathfrak{r}$. For $N \subseteq M$ we put

$$I(N) = \bigcap_{\mathfrak{m} \in N} \mathfrak{m}.$$

We shall denote by $\overline{N}$ the set of such $\mathfrak{m} \in M$ that $\mathfrak{m} \supseteq I(N)$. Then $I(N) = I(\overline{N})$. It will be proved (Theorem 2) that so defined closure operator permits us to introduce a topology in M. We shall call M *a structure space of* $\mathfrak{r}$. Of course, it is a subspace of Jacobson's structure space [3].

We denote by $\mathfrak{r}_\mathfrak{m}$ $(\mathfrak{m} \in M)$ the ring $\mathfrak{r}/\mathfrak{m}$, and by $h_\mathfrak{m}$ — the homomorphism of $\mathfrak{r}$ onto $\mathfrak{r}_\mathfrak{m}$. Let us denote by $\sigma(N)$ the complete direct sum of rings $\mathfrak{r}_\mathfrak{m}$, where $\mathfrak{m} \in N \subseteq M$. Then each element $x \in \sigma(N)$ can be regarded as a function $x(\mathfrak{m})$ defined for $\mathfrak{m} \in N$, and such that $x(\mathfrak{m}) \in \mathfrak{r}_\mathfrak{m}$.

Definition 1. The semi-simple ring $\mathfrak{r}$ is subdirectly imbedded into $\sigma(N)$, if there exists a subring $\sigma' \subseteq \sigma(N)$ isomorphic with $\mathfrak{r}$, and for each $\mathfrak{m} \in N$ and $r \in \mathfrak{r}$ there exists such $x' \in \sigma'$ that $x'(\mathfrak{m}) = r$.

THEOREM 1. *Let $\mathfrak{r}$ be a semi-simple ring and let M be its structure space. Then $\mathfrak{r}$ can be subdirectly imbedded into $\sigma(N)$, where $N \subseteq M$, if and only if $I(N) = 0$ (i.e. $\overline{N} = M$).*

*) We shall consider only two-sided ideals. An ideal $\mathfrak{m}$ of $\mathfrak{r}$ is called maximal modular, if $\mathfrak{r}/\mathfrak{m}$ is a simple ring with unity.

Proof. To each $x \in \mathfrak{r}$ we assign the function $x(\mathfrak{m})$ defined for $\mathfrak{m} \in N$ in the following way

$$x(\mathfrak{m}) = h_{\mathfrak{m}}(x) \in \mathfrak{r}_{\mathfrak{m}}.$$

It is easy to see that the mapping $x \to x(\mathfrak{m})$ is a homomorphism of $\mathfrak{r}$ onto some subring σ' of $\sigma(N)$.

Let us assume that $I(N) = 0$. If there exists such $x \in \mathfrak{r}$, $x \neq 0$ that $x(\mathfrak{m}) = 0$ for $\mathfrak{m} \in N$, then the element x belongs to each $\mathfrak{m} \in N$, which is impossible, since $I(N) = 0$. Thus, $x \to x(\mathfrak{m})$ is an isomorphism between $\mathfrak{r}$ and σ'. By the fact that each $h_{\mathfrak{m}}$ maps $\mathfrak{r}$ onto $\mathfrak{r}_{\mathfrak{m}}$ we conclude that $\mathfrak{r}$ is subdirectly imbeded into $\sigma(N)$. Conversely, let $\mathfrak{r}$ be subdirectly imbedded into $\sigma(N)$. If $I(N) \neq 0$, then $x \in I(N)$, $x \neq 0$ and $x \to 0$, which is impossible since the mapping $x \to x(\mathfrak{m})$ establishes an isomorphism between $\mathfrak{r}$ and σ'.

Remark 1. By the isomorphism between $\mathfrak{r}$ and σ' we shall identify the element $x \in \mathfrak{r}$ with the function $x(\mathfrak{m})$ defined either on the whole space M, or (according to appropriate imbedding) on some dense subspace $N \subseteq M$. Thus, we have

Corollary 1. *If $N \subseteq M$ then $x \in I(N)$ if and only if $x(\mathfrak{m}) = 0$ for each $\mathfrak{m} \in N$.*

Definition 2. The ideal $\mathfrak{a}$ of a semi-simple ring $\mathfrak{r}$ will be called representable, if there exists such $N \subseteq M$ that $\mathfrak{a} = I(N)$.

Lemma 1. *The representable ideal $\mathfrak{a} = I(N)$ of $\mathfrak{r}$ can be subdirectly imbedded into $\sigma(M - \overline{N})$.*

Proof. Let us observe that the ideal $\mathfrak{m} \cap \mathfrak{a}$, where $\mathfrak{m} \in M - \overline{N}$, is maximal modular in $\mathfrak{a}$. Indeed, for $\mathfrak{m} \in M - \overline{N}$ we have $I(N) + \mathfrak{m} = \mathfrak{r}$ and

$$\mathfrak{a}/\mathfrak{a} \cap \mathfrak{m} \sim \mathfrak{a} + \mathfrak{m}/\mathfrak{m} \sim \mathfrak{r}/\mathfrak{m} \sim \mathfrak{r}_{\mathfrak{m}}.$$

The intersection of all $\mathfrak{a} \cap \mathfrak{m}$, where $\mathfrak{m} \in M - \overline{N}$, is $\mathfrak{a} \cap I(M - \overline{N}) = I(\overline{N}) \cap I(M - \overline{N}) = \mathbf{0}$. Thus, by Theorem 1, it follows that $\mathfrak{a}$ can be subdirectly imbedded into $\sigma(M - N)$.

Corollary 2. *Each ideal $\mathfrak{m} \in M$ can be subdirectly imbedded into $\sigma(M - (\mathfrak{m}))$.*

Theorem 2. *Let $\mathfrak{r}$ be a semi-simple ring and let M be its structure space. Then*

(i)	$\overline{N_1 \cup N_2} = \overline{N}_1 \cup \overline{N}_2$	*for every* $N_1, N_2 \subseteq M$,
(ii)	$\overline{(\mathfrak{m})} = (\mathfrak{m})$	*for every* $\mathfrak{m} \in M$,
(iii)	$\overline{\emptyset} = \emptyset$	*where $\emptyset$ is an empty set,*
(iv)	$N = \overline{N}$	*for every* $N \subseteq M$.

Proof. Le us assume that $\mathfrak{m} \in N_1 \cup \overline{N}_2$. Then $\mathfrak{m} \supseteq I(N_1 \cup N_2) = I(N_1) \cap I(N_2)$. By Corollary 1, we have $x(\mathfrak{m}) = 0$ for each $x \in I(N_1) \cap I(N_2)$. Let us suppose that $\mathfrak{m} \in \overline{N}_1$ and $\mathfrak{m} \in \overline{N}_2$ i. e. $\mathfrak{m} \in M - \overline{N}_1$ and $\mathfrak{m} \in M - \overline{N}_2$.

By Lemma 1 applied to $I(N_1)$ and to $I(N_2)$ we obtain that there exists such $y \in I(N_1)$ and such $z \in I(N_2)$ that $y(\mathfrak{m}) = z(\mathfrak{m}) = 1_{\mathfrak{m}}$ (by $1_{\mathfrak{m}}$ we denote the unity element of $\mathfrak{r}_{\mathfrak{m}}$). Then $x(\mathfrak{m}) = y(\mathfrak{m})\, z(\mathfrak{m}) = 1_{\mathfrak{m}} \neq 0$ and $x \in I(N_1) \cap I(N_2)$, which concludes the contradiction. The converse inclusion and the properties (ii), (iii), (iv) are obvious.

Definition 3. (Andrunakijevič [1]). The annihilator $\mathfrak{a}^*$ of an ideal $\mathfrak{a}$ of $\mathfrak{r}$ is the set of such $x \in \mathfrak{r}$ that $x\mathfrak{a} = \mathfrak{a}x = 0$.

LEMMA 2. $[I(N)]^* = I(M - \overline{N})$, *where* $N \subseteq M$.

Proof. Let us suppose that $x \in [I(N)]^*$ and $x \in I(M - \overline{N})$. Then, by Corollary 1, there exists such $\mathfrak{m} \in M - \overline{N}$ that $x(\mathfrak{m}) \neq 0$. On the other hand, by Lemma 1, there exists such $y \in I(N)$ that $y(\mathfrak{m}) = 1_{\mathfrak{m}} \in \mathfrak{r}_{\mathfrak{m}}$. Then $xy \neq 0$, which is impossible since $x \in [I(N)]^*$. Conversely, if $x \in I(M - \overline{N})$, then for each $y \in I(N)$ we have $xy \in I(M - \overline{N}) \cap I(\overline{N}) = 0$. Thus $x \in [I(N)]^*$.

COROLLARY 3. $\mathfrak{m}^* = I(M - (\mathfrak{m}))$ for $\mathfrak{m} \in M$.

THEOREM 3. *Let* $\mathfrak{r}$ *be a semi-simple ring and let* M *be its structure space and let* $N \subseteq M$. *Then* $[I(N)]^{**} = I(N)$ *if and only if* $\overline{N}$ *is a closed domain in* M, *i.e.* $N = \overline{\operatorname{int}(N)}$, *where* $\operatorname{int}(\overline{N}) = M - \overline{M - \overline{N}}$.

Proof. Let us assume that $\overline{N} = \overline{\operatorname{int} N}$. Then, by Lemma 2, $[I(N)]^* = I(M - \overline{N})$ and $[I(N)]^{**} = I(M - M - \overline{N}) = I(\operatorname{int}(\overline{N})) = I(\overline{\operatorname{int}(\overline{N})}) = I(\overline{N})$. Conversely, if $[I(N)]^{**} = I(N)$ then, applying Lemma 2 again, we have $I(\operatorname{int}(\overline{N})) = I(N)$ and $\overline{\operatorname{int} \overline{N}} = \overline{N}$.

Definition 4. (Andrunakijevič [1]). A semi-simple ring $\mathfrak{r}$ will be called strongly semi-simple if each of its ideals is representable.

THEOREM 4. *Let* $\mathfrak{r}$ *be a strongly semi-simple ring and let* M *be its structure space. Then* $I(\operatorname{Fr}(N)) = I(N) \dotplus [I(N)]^*$, *where* $N \subseteq M$ *and* $\operatorname{Fr} \overline{N} = \overline{N} \cap \overline{M - \overline{N}}$

Proof. From the fact that $\mathfrak{r}$ is strongly semi-simple it follows that $\mathfrak{b} = I(N) + [I(N)]^* = I(N_1)$, where $N_1 \subseteq M$. Let us assume that $\mathfrak{m} \in \overline{N}_1$. Then $\mathfrak{m} \supseteq I(N)$ and $\mathfrak{m} \supseteq [I(N)]^* = I(M - \overline{N})$. Thus $\mathfrak{m} \in \overline{N} \cap \overline{M - \overline{N}} = \operatorname{Fr}(\overline{N})$. Conversely, let $\mathfrak{m} \in \overline{N} \cap \overline{M - \overline{N}}$. Then $\mathfrak{m} \supseteq I(N)$, since $\mathfrak{m} \in \overline{N}$, and, analogically, $\mathfrak{m} \supseteq I(M - \overline{N})$ since $\mathfrak{m} \in \overline{M - \overline{N}}$. Thus $\mathfrak{m} \supseteq \mathfrak{b}$ and $\mathfrak{m} \in \overline{N}_1$.

COROLLARY 4. *Let* $\mathfrak{a} = I(N)$ *be an ideal of a strongly semi-simple ring* $\mathfrak{r}$. *Then* $\mathfrak{r}$ *is a direct sum of* $\mathfrak{a}$ *and* $\mathfrak{a}^*$ *if and only if* $\overline{N}$ *is an open set in* M (*i.e. if* $\operatorname{Fr}(\overline{N}) = \emptyset$).

Using Zorn Lemma, the analogical fact can be easily proved for representable ideals of a semi-simple ring with unity.

LEMMA 3. *If* $N \subseteq M$, *then for each* $\mathfrak{m} \in M - \overline{N}$ *we have* $[\mathfrak{m} \cap I(N)]^* \cap \cap I(N) = \mathfrak{m}^*$.

Proof. By Lemma 2, we have $[\mathfrak{m} \cap I(N)]^* = [I(N \cup (\mathfrak{m}))]^* = I(M - \overline{N \cup (\mathfrak{m})})$. But, on the other hand, by Corollary 3 we obtain $[\mathfrak{m} \cap I(N)]^* \cap I(N) =$

$= I(M - \overline{N \cup (\mathfrak{m})}) \cap I(\overline{N}) = I(M - (\mathfrak{m})) = \mathfrak{m}^*$. because $(M - \overline{N \cup (\mathfrak{m})}) \cup \overline{N} = M - (\mathfrak{m})$.

LEMMA 4. *If* $\mathfrak{m} \in M$, *then either* $\mathfrak{m}^* = 0$ *or* $\mathfrak{m}^*$ *is isomorphic with* $\mathfrak{r}_\mathfrak{m}$.

Proof. Let us assign to each $x \in \mathfrak{m}^*$ the element $x(\mathfrak{m}) \in \mathfrak{r}_\mathfrak{m}$. This correspondence establishes an isomorphism between $\mathfrak{m}^*$ and some (not necessarily non-zero) subring $\mathfrak{k}$ of $\mathfrak{r}_\mathfrak{m}$. We shall prove that $\mathfrak{k}$ is an ideal of $\mathfrak{r}_\mathfrak{m}$. Indeed, let $k \in \mathfrak{k}$ and $r \in \mathfrak{r}_\mathfrak{m}$. Then there exists such $x \in \mathfrak{m}^*$ and such $y \in \mathfrak{r}$ that $x(\mathfrak{m}) = k$ and $y(\mathfrak{m}) = r$. But, on the other hand, $xy \in \mathfrak{m}^*$ and $xy(\mathfrak{m}) = x(\mathfrak{m})\, y(\mathfrak{m}) = kr \in \mathfrak{k} \subseteq \mathfrak{r}_\mathfrak{m}$. By the simplicity of $\mathfrak{r}_\mathfrak{m}$ we obtain: either $\mathfrak{m}^* \sim \mathfrak{k} = 0$, or $\mathfrak{m}^* \sim \mathfrak{k} = \mathfrak{r}_\mathfrak{m}$.

COROLLARY 5. *If* $\mathfrak{a}$ *is a simple ideal with unity (i.e. the ideal which as a ring is simple and has a unity element) of a semi-simple ring* $\mathfrak{r}$, *then there exists such* $\mathfrak{m} \in M$ *that* $\mathfrak{a} = \mathfrak{m}^* \sim \mathfrak{r}_\mathfrak{m}$.

Proof. The ideal $\mathfrak{a}$ is a direct summand of $\mathfrak{r}$, i. e. $\mathfrak{r} = \mathfrak{a} \dot{+} \mathfrak{m}$, where $\mathfrak{m} \in M$. Then, by Lemma 4, $\mathfrak{a} = \mathfrak{m}^* \sim \mathfrak{r}_\mathfrak{m}$.

2. Definition 5. The intersection D of all $N \subseteq M$ such that $I(N) = 0$ will be called *the minimal set of* $\mathfrak{r}$.

LEMMA 5. *The maximal modular ideal* $\mathfrak{m}$ *belongs to* D *if and only if* $\mathfrak{m}^* \neq 0$.

Proof. Let us assume that $\mathfrak{m} \in D$. If $\mathfrak{m}^* = 0$ then, by Corollary 3, $I(M - (\mathfrak{m})) = 0$ and $M - (\mathfrak{m}) \supseteq D$. Hence $\mathfrak{m} \in D$ and we have arrived at a contradiction. Conversely, let $\mathfrak{m} \notin D$. Then there exists such $N \subseteq M$ that $\mathfrak{m} \in N$ and $I(N) = 0$. But, on the other hand, $N \subseteq M - (\mathfrak{m})$ and $\mathfrak{m}^* = I(M - (\mathfrak{m})) = 0$.

Definition 6. A semi-simple ring $\mathfrak{r}$ will be called *special* if $I(D) = 0$. If, however, $D = \emptyset$ (i. e. $I(D) = \mathfrak{r}$), then $\mathfrak{r}$ will be called *completely non-special.*

The usual definition ([1] and [4]) of a special subdirect sum is expressed in terms of imbedding of $\mathfrak{r}$. The Definition 6 is formulated independently of a subdirect imbedding, thus we can speak of the properties of $\mathfrak{r}$ itself. From Theorem 1 and Lemma 5 it follows immediately that the semi-simple ring $\mathfrak{r}$ is special if and only if there exists a special subdirect imbedding of $\mathfrak{r}$ into a complete direct sum of simple rings with unity.

THEOREM 5. *Let* $\mathfrak{a}$ *be a representable ideal of a semi-simple ring* $\mathfrak{r}$. *Then* $\mathfrak{a}$ *is special (as a ring) if and only if* $\mathfrak{a} \cap I(D) = 0$; *and completely non-special if and only if* $\mathfrak{a} \cap I(D) = \mathfrak{a}$.

Proof. Let us assume that $\mathfrak{a} = I(N)$, $N \subseteq M$. For each $\mathfrak{m} \in M - \overline{N}$ the ideals $\mathfrak{a} \cap \mathfrak{m}$ are maximal modular in $\mathfrak{a}$ (see the proof of Lemma 1). Let $D_\mathfrak{a}$ be a minimal set of $\mathfrak{a}$. Then, by Lemmas 3 and 5, $\mathfrak{a} \cap \mathfrak{m} \in D$ if and only if $\mathfrak{m} \in D \cap (M - \overline{N})$. Now, let $I_\mathfrak{a}(D_\mathfrak{a})$ be the intersection of all $\mathfrak{a} \cap \mathfrak{m} \in D_\mathfrak{a}$. Then $I_\mathfrak{a}(D_\mathfrak{a}) = \mathfrak{a} \cap I(D \cap (M - \overline{N})) = I(\overline{N}) \cap I(D \cap (M - \overline{N})) = I(\overline{N} \cup (D \cap (M - \overline{N}))) = I(\overline{N} \cup D) = \mathfrak{a} \cup I(D)$, since $\overline{N} \cap (D \cap (M - \overline{N})) = \overline{N} \cup D$. Applying the Definition 6 we complete the proof.

THEOREM 6. (McCoy [4]). *A semi-simple ring* $\mathfrak{r}$ *is special if and only if each of its non-zero ideals contains a simple subideal with unity.*

Proof. Let us assume that each nonzero ideal of $\mathfrak{r}$ contains a simple subideal with unity. If $\mathfrak{r}$ is not special, then $I(D) \neq 0$. Let $\mathfrak{b}$ be a simple ideal with unity of $I(D)$. It is easy to verify (see e.g. [5]) that $\mathfrak{b}$ is also an ideal of $\mathfrak{r}$. Thus, by Corollaries 3 and 5, there exists such $\mathfrak{m} \in M$ that $\mathfrak{m}^* = I(M-(\mathfrak{m})) = \mathfrak{b} \subseteq I(D)$. From $I(M-(\mathfrak{m})) \subseteq I(D)$ we obtain $M-(\mathfrak{m}) \supseteq D$. But, on the other hand, Lemma 5 gives us $\mathfrak{m} \in D$, since $\mathfrak{m}^* = \mathfrak{b} \neq 0$. (Contradiction!). Conversely, let $\mathfrak{r}$ be a special ring and let $\mathfrak{a}$ be one of its ideals. Then $I(D) = 0$ and, by Theorem 1, $\mathfrak{r}$ can be subdirectly imbedded into the complete direct sum $\sigma(D)$. Then, for each $x \in \mathfrak{a}$, $x \neq 0$ there exists such $\mathfrak{m} \in D$ that $x(\mathfrak{m}) \neq 0$. By Lemmas 4 and 5 we have $\mathfrak{m}^* \sim \mathfrak{r}_\mathfrak{m}$ for each $\mathfrak{m} \in D$. Let y be such element of $\mathfrak{m}^* \subseteq \mathfrak{r}$ that $y(\mathfrak{m}) = 1_\mathfrak{m} \in \mathfrak{r}_\mathfrak{m}$. Then $\mathfrak{m}^* \cap \mathfrak{a} \neq 0$ since $xy \in \mathfrak{m}^* \cap \mathfrak{a}$ and $xy \neq 0$. By the simplicity of $\mathfrak{m}^*$ we have $\mathfrak{a} \supseteq \mathfrak{m}^* \sim \mathfrak{r}_\mathfrak{m}$.

Now, we shall prove the analogous theorem on the completely non-special rings.

THEOREM 7. *A semi-simple ring* $\mathfrak{r}$ *is completely non-special if and only if* $\mathfrak{r}$ *contains no simple ideals with unity.*

Proof. Let $\mathfrak{r}$ be a completely non-special ring. If $\mathfrak{r}$ contains a simple ideal $\mathfrak{a}$ with unity, then, by Corollary 5, there exists such $\mathfrak{m} \in M$ that $\mathfrak{a} = \mathfrak{m}^* \sim \mathfrak{r}_\mathfrak{m}$. Thus, by Lemma 5, $\mathfrak{m} \in D$, which is impossible, since $D = \emptyset$. Conversely, let us assume that $\mathfrak{r}$ is not a completely non-special ring and let $\mathfrak{m} \in D \neq \emptyset$. Then, by Lemmas 4 and 5, $\mathfrak{m}^* \sim \mathfrak{r}_\mathfrak{m}$ is a simple ideal of $\mathfrak{r}$ with unity.

COROLLARY 6. *A semi-simple ring is completely non-special if and only if the annihilator of each of its maximal modular ideals is zero.*

It follows in particular that the ring of integers is completely non-special.

3. LEMMA 6. $I(\operatorname{int}(\overline{D}) \cup (M-D)) = 0$.

Proof. By the additivity of closure operation (Theorem 2), we have $\overline{\operatorname{int}(\overline{D}) \cup (M-\overline{D})} = \overline{M-\overline{M-D} \cup (M-\overline{D})} \supseteq M-\overline{M-D} \cup \overline{(M-\overline{D})} = M$.

LEMMA 7. $\operatorname{int}(\overline{D}) \supseteq D$.

Proof. By Lemma 6 and Definition 5, we have $\operatorname{int}(\overline{D}) \cup (M-\overline{D}) \supseteq D$. But, on the other hand, $(M-\overline{D}) \cap D \subseteq (M-\overline{D}) \cap \overline{D} = \emptyset$. Thus, $\operatorname{int}(\overline{D}) \supseteq D$.

THEOREM 8. *The closure* $\overline{D}$ *of the minimal set of a semi-simple ring* $\mathfrak{r}$ *is a closed domain in the structure space* M.

Proof. By Lemma 7, we have $\overline{\operatorname{int}(\overline{D})} \supseteq \overline{D}$. But, on the other hand, $\operatorname{int}(\overline{D}) \subseteq \overline{D}$ and $\overline{\operatorname{int}(\overline{D})} \subseteq \overline{D}$. Thus, $\overline{\operatorname{int}(\overline{D})} = \overline{D}$.

Definition 7. Let $\mathfrak{r}$ be a semi-simple ring, let M be a structure space of $\mathfrak{r}$ and let D be the minimal set of $\mathfrak{r}$. We shall call the ideal

$s(\mathfrak{r}) = I(M - \overline{D})$ the special part, and the ideal $n(\mathfrak{r}) = I(D)$ — the completely non-special part of $\mathfrak{r}$.

By Theorems 3 and 8, we have.

COROLLARY 7. *$[n(\mathfrak{r})]^{**} = n(\mathfrak{r})$ and $[n(\mathfrak{r})]^{*} = s(\mathfrak{r})$.*

COROLLARY 8. *$[s(\mathfrak{r})]^{**} = s(\mathfrak{r})$ and $[s(\mathfrak{r})]^{*} = n(\mathfrak{r})$.*

Proof. By Lemma 2 and Theorem 8, we have $[s(\mathfrak{r})]^{*} = [I(M - \overline{D})]^{*} = I(M - \overline{M - \overline{D}}) = I(\operatorname{int}(\overline{D})) = I(\operatorname{int}(\overline{D})) = I(\overline{D})$. Applying Lemma 2 again, we have $[s(\mathfrak{r})]^{**} = I(\overline{D}) = I(M - \overline{D}) = s(\mathfrak{r})$.

THEOREM 9. *Let $\mathfrak{r}$ be a semi-simple ring. Then $s(\mathfrak{r})$ (as a ring) is special and $n(\mathfrak{r})$ is completely non-special.*

Proof. Let us observe that $s(\mathfrak{r}) \cap I(D) = I(M - \overline{D}) \cap I(\overline{D}) = I(M) = 0$. Thus, by Theorem 5, $s(\mathfrak{r})$ is special.

Further, $n(\mathfrak{r}) \cap I(D) = I(D) \cap I(D) = I(D) = n(\mathfrak{r})$. Thus, again by Theorem 5, $n(\mathfrak{r})$ is completely non-special.

THEOREM 10. *Let $\mathfrak{a}$ be a representable ideal of a semi-simple ring $\mathfrak{r}$. If $\mathfrak{a}$ (as a ring) is special then $\mathfrak{a} \subseteq s(\mathfrak{r})$, if, however, $\mathfrak{a}$ is completely non-special then $\mathfrak{a} \subseteq n(\mathfrak{r})$.*

Proof. Let us assume that $\mathfrak{a} = I(N)$, $N \subseteq M$. If $\mathfrak{a}$ is special, then, by Theorem 5, $\mathfrak{a} \cap I(D) = 0$. Then for each $x \in \mathfrak{a}$, $y \in I(D)$ we have $xy \in \mathfrak{a} \cap I(D) = 0$. Thus, by Lemma 2, $\mathfrak{a} \subseteq [I(D)]^{*} = I(M - \overline{D}) = s(\mathfrak{r})$.

Now, let us assume that $\mathfrak{a}$ is completely non-special, Then, again by theorem 5, $\mathfrak{a} \cap I(D) = \mathfrak{a}$. Thus $\mathfrak{a} \subseteq I(D) = n(\mathfrak{r})$.

Finally, let us observe that by Corollaries 4 and 7, the following result can be obtained.

THEOREM 11. *Let $\mathfrak{r}$ be a strongly semi-simple ring. Then $\mathfrak{r}$ is a direct sum of its special and completely non-special part if and only if the closure D of the minimal set of $\mathfrak{r}$ is open in the structure space M* (i.e. *if* $\operatorname{Fr}(\overline{D}) = \emptyset$).

The same fact can be easily proved for semi-simple rings with unity.

The author is sincerely indebted to Professor Łoś and Dr Sąsiada for their remarks and suggestions.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VIII, No. 4, 1960

A Possible Mechanism of the High Energy Maxima in Pion-Nucleon Cross Sections

by

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Presented by W. RUBINOWICZ on February 19, 1960

In this note the motion of the core inside the nucleon is pointed out as a possible reason of three high energy maxima observed in $\pi^- p$ and $\pi^+ p$ scatterings [1], [2]. A suggestion is made that these peaks are resonances (probably $D_{3/2\,1\,2}$, $D_{5/2\,1\,2}$, $D_{5\,2\,3/2}$) provided by a coupling of slowly moving core inside the nucleon to the pion field:

$$H^{N\pi} = \frac{f_0}{\mu} \vec{\sigma}\tau_a \int d_3 x\, \varrho(|\vec{x}_0 - \vec{x}|)\, \vec{\partial}\, \varphi_a(\vec{x}), \tag{1}$$

where $\vec{x}_0$ is the position of the centre of mass of the core. This possible mechanism of higher excitations of the nucleon would compete with a mechanism *via* pion-pion interaction suggested previously [3]—[6] (see also [7]).

Making an opportunistic assumption that recoils of the centre of mass of the nucleon, as a whole, are negligible in the transition amplitude, we put

nucleon momentum $(\vec{p}) \equiv$ core momentum $(\vec{p}_0)$ + cloud momentum $= 0$

and derive for D-wave elastic scattering the following partial dispersion relations *)

$$g_{JT}(\omega) = \frac{1}{(2\pi)^2} \int_{\mu}^{\infty} d\omega'\, k'^3\, v^2(k') \left[\frac{\sigma_{JT}(\omega')}{\omega' - (\omega + i\epsilon)} + \sum_{J'T'} A_{JJ'} B_{TT'} \frac{\sigma_{J'T'}(\omega')}{\omega' + \omega} \right], \tag{2}$$

where

$$(A_{JJ'}) = \frac{1}{5} \begin{pmatrix} -1 & 6 \\ 4 & 1 \end{pmatrix} \quad \text{and} \quad (B_{TT'}) = \frac{1}{3} \begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix}. \tag{3}$$

*) We apply in (1) a kind of multipole expansion $\exp(i\vec{k}\vec{x}_0) = 1 + i\vec{k}\vec{x}_0 + \ldots$ and approximate each partial wave retaining in it only the lowest contributing power of $k\,x_0$.

Here, we use the notations

$$(4)\qquad g_{JT}(\omega)=\frac{1}{k^5\,v^2(k)}\sin\eta_{JT}(\omega)\,e^{i\eta_{JT}(\omega)},\qquad \sigma_{JT}(\omega)=4\pi k^3\,v^2(k)\,\mathrm{Im}\;g_{JT}(\omega),$$

where η_{JT} are D-wave (complex) phase shifts. Let us note that

$$(5)\qquad \cot\eta_{JT}(\omega)=\frac{1}{k^5\,v^2(k)}\,\frac{1}{g_{JT}(\omega)}+i.$$

The form (3) of $(A_{J,J'})$ and $(B_{TT'})$ follows from the "crossing" relations for D-wave projection operators. Vanishing of inhomogeneous terms in (2) is a more general property of D-waves in the zero-recoil limit [8].

When applying an "effective range" approach of Chew-Low type [10], we get

$$(6)\qquad \mathrm{Re}\,\frac{g_{JT}(0)}{g_{JT}(\omega)}\approx 1-r_{JT}\omega \;\text{ and }\; k^5\,v^2(k)\,g_{JT}(0)\,\mathrm{Re}\cot\eta_{JT}(\omega)\approx 1-r_{JT}\omega,$$

where

$$(7)\qquad r_{JT}=\frac{1}{(2\pi)^2\,g_{JT}(0)}\int_{\mu}^{\infty}d\omega\;\frac{1}{\omega^2 k^3\,v^2(k)}\left[\sigma_{JT}(\omega)-\sum_{J'T'}A_{JJ'}\,B_{TT'}\,\sigma_{J'T'}(\omega)\right]$$

and

$$(8)\qquad g_{JT}(0)=\frac{1}{(2\pi)^2}\int_{\mu}^{\infty}d\omega\;\frac{1}{\omega k^3\,v^2(k)}\left[\sigma_{JT}(\omega)+\sum_{J'T'}A_{JJ'}\,B_{TT'}\,\sigma_{J'T'}(\omega)\right].$$

It follows from (7) and (3) that

$$(9)\qquad g_{33}(0)\,r_{33}=-5\,g_{31}(0)\,r_{31}-9\,g_{53}(0)\,r_{53},\quad g_{51}(0)\,r_{51}=6\,g_{31}(0)\,r_{31}+10\,g_{53}(0)\,r_{53}$$

(we write as usually $2J\,2T$ instead of JT). Rewriting (8) and (7) more explicitly, we have

$$(10)\qquad \begin{pmatrix} g_{31}(0)\\ g_{33}(0)\\ g_{51}(0)\\ g_{53}(0)\end{pmatrix}=\begin{pmatrix} 16 & -4 & -6 & 24\\ -2 & 14 & 12 & 6\\ -4 & 16 & 14 & 4\\ 8 & 4 & 2 & 16\end{pmatrix}\begin{pmatrix} I_{31}\\ I_{33}\\ I_{51}\\ I_{53}\end{pmatrix},$$

$$\text{where}\qquad I_{JT}=\frac{1}{15}\,\frac{1}{(2\pi)^2}\int_{\mu}^{\infty}d\omega\,\frac{\sigma_{JT}(\omega)}{\omega k^3\,v^2(k)}>0$$

and

$$(11)\qquad \begin{pmatrix} g_{31}(0)\,r_{31}\\ g_{33}(0)\,r_{33}\\ g_{51}(0)\,r_{51}\\ g_{53}(0)\,r_{53}\end{pmatrix}=\begin{pmatrix} 14 & 4 & 6 & -24\\ 2 & 16 & -12 & -6\\ 4 & -16 & 16 & -4\\ -8 & -4 & -2 & 14\end{pmatrix}\begin{pmatrix} J_{31}\\ J_{33}\\ J_{51}\\ J_{53}\end{pmatrix},$$

$$\text{where}\qquad J_{JT}=\frac{1}{15}\,\frac{1}{(2\pi)^2}\int_{\mu}^{\infty}d\omega\,\frac{\sigma_{JT}(\omega)}{\omega^2 k^3\,v^2(k)}>0\,.$$

It is probable from the point of view of present experimental data that the observed high energy peaks are due to $D_{3\,2\,1\,2}$, $D_{5\,2\,1/2}$ and $D_{5/2\,3\,2}$ resonant states [1], [2]. In this case it seems to follow from (10) that all g_{JT} (0) are positive. Then, we can see from (9) that r_{33} and r_{51} are negative and positive, respectively, if only r_{31} and r_{53} given by (11) are positive. The last possibility is not in contradiction with the experimental data. In such a case we should have

$$r_{31} > 0\,, \quad r_{33} < 0\,, \quad r_{51} > 0\,, \quad r_{53} > 0\,. \tag{12}$$

A more certain and quantitative conclusion could be drawn, if results of a phase shift analysis were available.

If inequalities (12) were true, we could expect that the theory would lead to resonances in $D_{3\,2\,1/2}$, $D_{5\,2\,1\,2}$, $D_{5\,2\,3\,2}$ states (beside the well known resonance in the $P_{3/2\,3/2}$ state).

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VIII, No. 4, 1960

On the Conservation Law of the Angular Momentum of a System of Rotating Bodies in the General Theory of Relativity

by

R. MICHALSKA

Presented by L. INFELD on February 25, 1960

1. The relativistic equations of the translatory motion of two spherically symmetric rotating bodies have been derived by Tulczyjew [1]. Due to the post-Newtonian rotational terms in Tulczyjew's equations, the orbital angular momentum of the system of bodies is not conserved. Therefore, if the conservation law of the angular momentum is valid in this case, the internal angular momenta must be functions of time. It is the aim of the present note to study this time dependence and to derive an explicit expression for the conserved total angular momentum of the system.

2. Following Tulczyjew's paper we assume a singular energy-momentum density tensor of the form

$$T^{\alpha\beta} = \sum_{p=1}^{2} \{ \overset{p}{t}{}^{\alpha\beta}(t)\, \delta(x^i - \overset{p}{\xi}{}^i(t)) - \overset{p}{t}{}^{r\alpha\beta}(t)\, \delta_{,r}(x^i - \overset{p}{\xi}{}^i(t)) \}, \tag{1}$$

wherein δ are the modified [2] Dirac δ functions and $\overset{p}{\xi}{}^i$ denote the spatial co-ordinates of the p-th singularity.

Evaluating $T^{\alpha\beta}_{;\beta}$, we obtain the following equations

$$\overset{1}{t}{}^{\alpha 0}{}_{,0} + \overset{1}{t}{}^{00}\left\{\begin{matrix}\overset{1}{\widetilde{\alpha}}\\ 0\,0\end{matrix}\right\} + 2\,\overset{1}{t}{}^{0r}\left\{\begin{matrix}\overset{1}{\widetilde{\alpha}}\\ 0r\end{matrix}\right\} + \overset{1}{t}{}^{rs}\left\{\begin{matrix}\overset{1}{\widetilde{\alpha}}\\ rs\end{matrix}\right\} + \overset{1}{t}{}^{r00}\left\{\begin{matrix}\overset{1}{\widetilde{\alpha}}\\ 0\,0\end{matrix}\right\}_{,r} + \tag{2a}$$

$$+ 2\,\overset{1}{t}{}^{rs0}\left\{\begin{matrix}\overset{1}{\widetilde{\alpha}}\\ s0\end{matrix}\right\}_{,r} + \overset{1}{t}{}^{rst}\left\{\begin{matrix}\overset{1}{\widetilde{\alpha}}\\ st\end{matrix}\right\}_{,r} = 0,$$

$$\overset{1}{t}{}^{r\alpha 0}{}_{,0} + \overset{1}{t}{}^{\alpha 0}\,\overset{1}{\xi}{}^{r}{}_{,0} - \overset{1}{t}{}^{\alpha r} + \overset{1}{t}{}^{r00}\left\{\begin{matrix}\overset{1}{\widetilde{\alpha}}\\ 0\,0\end{matrix}\right\} + 2\,\overset{1}{t}{}^{rs0}\left\{\begin{matrix}\overset{1}{\widetilde{\alpha}}\\ s0\end{matrix}\right\} + \overset{1}{t}{}^{rst}\left\{\begin{matrix}\overset{1}{\widetilde{\alpha}}\\ st\end{matrix}\right\} = 0, \tag{2b}$$

$$(2c) \qquad \overset{1}{t}{}^{r\alpha 0}\,\overset{1}{\xi}{}^{s}_{,0} + \overset{1}{t}{}^{s\alpha 0}\,\overset{1}{\xi}{}^{r}_{,0} - \overset{1}{t}{}^{r\alpha s} - \overset{1}{t}{}^{s\alpha r} = 0$$

for the first body and similar equations for the second one. The symbol $\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} = \frac{1}{2} g^{\alpha\delta}(g_{\beta\delta,\gamma} + g_{\gamma\delta,\beta} - g_{\beta\gamma,\delta})$ denotes the Christoffel symbol of the second kind and $\left\{ \begin{matrix} \overset{p}{\widetilde{\alpha}} \\ \beta\gamma \end{matrix} \right\} = \int\limits_{(p)} \delta(x^i - \overset{p}{\xi}{}^i) \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} d_3 x$.

Eqs. (2), (in number of 40 for each singularity), contain 40 unknown functions $t^{\alpha\beta}(t)$ and $t^{r\alpha\beta}(t)$. Some additional information about $T^{\alpha\beta}$ is, therefore, needed in order to derive from (2) the equations of motion, i.e. equations for $\overset{p}{\xi}{}^k(t)$. We may obtain such information from the postulate, that a correspondence between (1) and the tensor of continuous matter exists. It can be shown [3] that, if $\overset{p}{\xi}{}^i$ have to correspond to the centres of mass of the bodies of finite size, the coefficients $\overset{p}{t}{}^{r00}$, in the used approximation, must be

$$(3) \qquad \overset{p}{t}{}^{r00} = \frac{1}{2}\,\overset{p}{\xi}{}^{i}_{,0}\,\overset{p}{S}{}^{ri},$$

where $\overset{p}{S}{}^{ri}$ denotes the internal angular momentum of the p-th body.

The Eqs. (2a), (2b) for $\alpha = 0$, the symmetrical part of (2b) for $\alpha = k$, and (2c), together with the condition (3) and the field equations, determine the coefficients $t^{\alpha\beta}$ and $t^{r\alpha\beta}$ in the EIH approximation method. They lead to the following expressions for the first body

$$(4) \qquad \begin{cases} \overset{1}{t}{}^{00}_{2} = \overset{1}{m} = \text{const}; \\ \overset{1}{t}{}^{00}_{4} = \frac{1}{2}\,\overset{1}{m}\,\overset{1}{\xi}{}^{i}_{,0}\,\overset{1}{\xi}{}^{i}_{,0} + \overset{1}{m}\,\overset{2}{m}/r, \quad r = [(\overset{1}{\xi}{}^k - \overset{2}{\xi}{}^k)(\overset{1}{\xi}{}^k - \overset{2}{\xi}{}^k)]^{1/2}; \\ \overset{1}{t}{}^{0k}_{3} = \overset{1}{m}\,\overset{1}{\xi}{}^{k}_{,0}; \\ \overset{1}{t}{}^{0k}_{5} = \overset{1}{t}{}^{00}_{4}\,\overset{1}{\xi}{}^{k}_{,0} - \frac{1}{2}\,\overset{2}{m}\,\overset{1}{S}{}^{kj}\left(\frac{1}{r}\right)_{,\overset{1}{\xi}{}^j}; \\ \overset{1}{t}{}^{r0s}_{3} = \frac{1}{2}\,\overset{1}{S}{}^{rs}, \qquad \overset{1}{S}{}^{rs} = -\overset{1}{S}{}^{sr}; \\ \overset{1}{t}{}^{rst}_{4} = \frac{1}{2}(\overset{1}{S}{}^{rs}\,\overset{1}{\xi}{}^{t}_{,0} + \overset{1}{S}{}^{rt}\,\overset{1}{\xi}{}^{s}_{,0}), \qquad \overset{1}{t}{}^{rs} = \overset{1}{m}\,\overset{1}{\xi}{}^{r}_{,0}\,\overset{1}{\xi}{}^{s}_{,0} \end{cases}$$

and to similar expressions for the second one. The indices written below indicate the orders of the quantities in the EIH approximation.

Eqs. (2a) for $\alpha = k$ give the equations of the translatory motion, which are thoroughly studied in Tulczyjew's paper. One of the consequences of Eqs. (2a), which is of interest here, is that the orbital angular momentum of the bodies is not conserved. Namely, in the centre of mass system of two bodies we have

$$L^i_{,0} = 2\mu\left[v^i\left(\frac{1}{r}\right)_{,k} - v^k\left(\frac{1}{r}\right)_{,i}\right]S^k, \tag{5}$$

where

$$L^i = \mu\left(1 + \frac{1}{2}[1,-1]\,v^l v^l + [3,7]\,M/r\right)\varepsilon^{ijk}\,\xi^j\,v^k + 2\mu M\,\varepsilon^{ijk}\,\varepsilon^{krs}\left(\frac{1}{r}\right)_{,j}\xi^s\,S^r$$

is the orbital angular momentum, ε^{ijk} denotes the permutation symbol, $\xi^i = \overset{1}{\xi}{}^i - \overset{2}{\xi}{}^i$, $v^i = \xi^i_{,0}$ is the relative velocity,

$$M = \overset{1}{m} + \overset{2}{m}, \qquad \mu = \overset{1}{m}\,\overset{2}{m}/M, \qquad [u,w]\,M^2 = u\overset{1}{m}{}^2 + w\overset{1}{m}\,\overset{2}{m} + u\overset{2}{m}{}^2,$$

and

$$S^k = \left(1 + \frac{3}{4}\frac{\overset{2}{m}}{\overset{1}{m}}\right)\overset{1}{S}{}^k + \left(1 + \frac{3}{4}\frac{\overset{1}{m}}{\overset{2}{m}}\right)\overset{2}{S}{}^k, \tag{6}$$

$$\overset{p}{S}{}^k = \frac{1}{2}\,\epsilon^{kij}\,\overset{p}{S}{}^{ij}, \qquad p = 1,2\,.$$

3. Let us consider now Eqs. (2b) for $\alpha = k$. In the fourth order they show that the quantities $\overset{p}{\underset{3}{S}}{}^{rs}$, which can be interpreted as the Newtonian internal angular momenta of the bodies [1], are constant. In the sixth order Eqs. (2b) for $\alpha = k$ read

$$\overset{1}{\underset{5}{t}}{}^{i0k}{}_{,0} = \overset{1}{\underset{6}{t}}{}^{ik} - \overset{1}{\underset{5}{t}}{}^{k0}\,\overset{1}{\xi}{}^i_{,0} - \overset{1}{\underset{4}{t}}{}^{i00}\left\{\begin{matrix}\overset{1}{\tilde{k}}\\ 0\,0\end{matrix}\right\} - 2\,\overset{1}{t}{}^{ij0}\left\{\begin{matrix}\overset{1}{\tilde{k}}\\ j0\end{matrix}\right\} - \overset{1}{t}{}^{ijl}\left\{\begin{matrix}\overset{1}{\tilde{k}}\\ jl\end{matrix}\right\}. \tag{7}$$

Taking the antisymmetrical part of (7), and making use of (3) and (4), we obtain, in the centre of mass system,

$$\left(\overset{1}{\underset{5}{S}}{}^{ik} + \frac{\overset{2}{m}}{M}\,v^j\,v^j\,\overset{1}{\underset{3}{S}}{}^{ik}\right)_{,\,0\,1} = 2\mu\left(1 + \frac{3}{4}\frac{\overset{2}{m}}{\overset{1}{m}}\right)\left[v^j\left(S^{ij}\left(\frac{1}{r}\right)_{,k} + \right.\right. \tag{8}$$

$$\left.\left. - \overset{1}{S}{}^{kj}\left(\frac{1}{r}\right)_{,i}\right) + \left(\frac{1}{r}\right)_{,j}(\overset{1}{S}{}^{ij}\,v^k - \overset{1}{S}{}^{kj}\,v^i)\right],$$

where

$$\overset{1}{\underset{5}{S}}{}^{ik} = \overset{1}{\underset{5}{t}}{}^{i0k} - \overset{1}{\underset{5}{t}}{}^{k0i},$$

For the second body the same calculations give

$$\left(\overset{2}{\underset{5}{S}}{}^{ik} + \frac{\overset{1}{m}}{M} v^j v^j \overset{2}{\underset{3}{S}}{}^{ik}\right)_{,0} = 2\mu\left(1 + \frac{3}{4}\frac{\overset{1}{m}}{\overset{2}{m}}\right)\left[v^j\left(\overset{2}{S}{}^{ij}\left(\frac{1}{r}\right)_{,k} + \right.\right. \tag{9}$$

$$\left.\left. - \overset{2}{S}{}^{kj}\left(\frac{1}{r}\right)_{,i}\right) + \left(\frac{1}{r}\right)_{,j}(\overset{2}{S}{}^{ij} v^k - \overset{2}{S}{}^{kj} v^i)\right].$$

Therefore, we have

$$\left(\overset{1}{\underset{5}{S}}{}^{k} + \overset{2}{\underset{5}{S}}{}^{k} + \frac{1}{M}\left(\overset{2}{m}\overset{1}{\underset{3}{S}}{}^{k} + \overset{1}{m}\overset{2}{\underset{3}{S}}{}^{k}\right)\right)_{,0} = 2\mu S^i\left(v^i\left(\frac{1}{r}\right)_{,k} - v^k\left(\frac{1}{r}\right)_{,i}\right), \tag{10}$$

where $\overset{1}{S}{}^k$, $\overset{2}{S}{}^k$ and S^k are defined by (6).

Eqs. (5) and (10) give us finally a law of conservation

$$\frac{d}{dx^0} J^k = 0, \tag{11}$$

where

$$J^k = L^k + \overset{1}{S}{}^k + \overset{2}{S}{}^k + \frac{1}{M} v^i v^i (\overset{2}{m}\overset{1}{S}{}^k + \overset{1}{m}\overset{2}{S}{}^k). \tag{12}$$

Now, the quantities $\overset{p}{\underset{5}{S}}{}^k$ appear in Eqs. (2b) in addition to the Newtonian internal momenta $\overset{p}{\underset{3}{S}}{}^k$. This fact, together with the form (12) of the conserved quantity J^k, justify the interpretation of $\overset{p}{\underset{5}{S}}{}^k$ as the post-Newtonian internal angular momenta of the bodies.

Eq. (12) presents the rule of addition of the angular momenta in the presence of the "spin-orbit" interaction. The last term in (12) is connected with the non-linear character of the theory of gravitation.

The author wishes to express her gratitude to Professor L. Infeld for his kind interest in this work.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VIII, No. 4, 1960

Action Principle for the Motion of Rotating Bodies in the General Theory of Relativity

by

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Presented by L. INFELD on February 25, 1960

1. The problem of motion of rotating bodies of comparable masses in the general theory of relativity has been investigated by two methods. Fock [1] used the continuous energy-momentum tensor to describe the bodies of finite size with internal structure. Tulczyjew [2] represented the spherically symmetric rotating bodies by the singularities of the field. In both of these methods the post-Newtonian equations of the translatory motion derived from the field equations and the Lagrange-functions were found mechanically from the explicit form of the equations of motion *).

Another approach to the problem of motion in the general theory of relativity has been recently presented by Plebański and Bażański [4]. Starting from a general action principle, the authors have found the Fokker-type Lagrangian for the perfect fluid, i.e. a Lagrangian in which only dynamical variables of the fluid appear. A model of isolated drops of the perfect fluid was used in [4] to describe the motion of bodies without internal degrees of freedom. It is the aim of this paper to extend the Fokker-type Lagrangian method to the case of rotating bodies. This method will enable us to obtain the Lagrangian for rotating bodies from a general action principle and to derive the equations of the translatory and internal motion of the bodies by a procedure of variation.

Our assumptions are very close to those made by Fock [1] as far as the structure of the bodies is concerned. The consistent use of the approximation with respect to l/L **) and a procedure of renormalization allow us to write the Lagrangian in a relatively simple form. In a middle step of calculations we obtain the Lagrangian of Fock. Our final ex-

*) A more general method of derivation of Tulczyjew's Lagrangian is given in [3].

**) l is the length characterizing the dimensions of the bodies and L the distance between them.

pression is in agreement with Tulczyjew's results for spherically symmetric bodies. The terms in the Lagrangian which depend on the oblateness of the bodies indicate, that the "gravitational skeleton" of Mathisson [5] does not exist for the system of "rotating drops".

2. We start with the problem of motion of continuous matter and assume that the matter has the properties of the perfect fluid. That means: it is described by the scalar density $\varrho(x)$, the pressure $p(x)$, which is a single valued function of ϱ (barotropic fluid), and by the velocity field $u^\alpha = \frac{dx^\alpha}{ds}$, *), $u_\alpha u^\alpha = 1$. Let $g_{\alpha\beta}(x)$ denote the metric tensor. The equations of the fluid and the Einstein field equations can be derived [4] from the single action principle

$$\delta W = 0,$$

$$W = -c \int d_4 x \sqrt{-g}\left[\varrho + \frac{1}{c^2}\left(\varrho \int_0^\varrho \frac{dp}{\varrho} - p\right)\right] + \frac{c^3}{16\pi k}\int d_4 x \sqrt{-g}\, G,$$

where

$$G = g^{\mu\nu}\left(\begin{Bmatrix}\varrho\\ \nu\sigma\end{Bmatrix}\begin{Bmatrix}\sigma\\ \mu\varrho\end{Bmatrix} - \begin{Bmatrix}\varrho\\ \mu\nu\end{Bmatrix}\begin{Bmatrix}\sigma\\ \varrho\sigma\end{Bmatrix}\right),$$

k denotes the gravitational constant and c — the velocity of light.

The density $\varrho(x)$ satisfies the equation of continuity

$$(\varrho u^\alpha)_{;a} = 0. \tag{2}$$

Introducing a new quantity

$$\sigma(x) = \sqrt{-g}\,\varrho(x)\frac{dx^0}{ds}, \tag{3}$$

we can write Eq. (2) in the form

$$(\sigma v^\beta)_{,\beta} = 0, \qquad v^\beta = \frac{dx^\beta}{dx^0}. \tag{4}$$

The function $\sigma(x)$ represents the rest density measured in a locally Cartesian co-ordinate system and the mass $M = \int d_3 x\, \sigma(x)$ is an invariant conserved quantity.

Introducing the Lagrange function $\mathcal{L}$ we can write (1) in the form

$$W = \frac{c^3}{k}\int dx_0\, \mathcal{L}, \tag{5}$$

where

$$\mathcal{L} = -\frac{k}{c^2}\int d_3 x\, \sigma(x)\,(g_{\alpha\beta} v^\alpha v^\beta)^{1/2} - \tag{6}$$

$$-\frac{k}{c^4}\int d_3 x \sqrt{-g}\left(\varrho\int\frac{dp}{\varrho} - p\right) + \frac{1}{16\pi}\int d_3 x \sqrt{-g}\, G.$$

*) We use in this section the following notations:

$$ds = (g_{\alpha\beta}\, dx^\alpha\, dx^\beta)^{1/2}, \quad g = \det(g_{\alpha\beta}), \quad \begin{Bmatrix}\alpha\\ \beta\gamma\end{Bmatrix} = \tfrac{1}{2}\, g^{\alpha\delta}\,(g_{\beta\delta,\,\gamma} + g_{\gamma\delta,\,\beta} - g_{\beta\gamma,\,\delta}),$$

the Greek indices assume the values 0, 1, 2, 3, the Latin ones — 1, 2, 3.

Eliminating from (6) the gravitational field by means of the EIH method of approximations, Plebański and Bażański [4] have obtained the following functional depending on σ and v^k only

$$(7)\qquad \underset{\to 6}{\mathcal{L}} = -\int d_3x\,\mu + \frac{1}{2}\int d_3x\,\mu v^i v^i + \frac{1}{8}\int d_3x\,\mu (v^i v^i)^2 +$$

$$+\frac{3}{4}\int d_3x \int d_3x'\,\mu\mu' \frac{1}{|\underline{x}-\underline{x}'|}(v^i v^i + v'^i v'^i) -$$

$$-2\int d_3x \int d_3x'\,\mu\mu' v^i v^{i'} \frac{1}{x-\underline{x}'|} - \frac{1}{4}\int d_3x \int d_3x'\,\mu\mu' |\underline{x}-\underline{x}|_{,ik'} v^i v'^k +$$

$$+\frac{1}{2}\int d_3x \int d_3x'\,\mu\mu' \frac{1}{|\underline{x}-\underline{x}'|} - \frac{1}{6}\int d_3x \int d_3x' \int d_3x''\,\mu\mu'\mu''\,[\underline{x},\underline{x}',\underline{x}''] +$$

$$-\frac{k}{c^4}\int d_3x \left(\sigma \int_0^p \frac{dp}{\sigma} - p\right) + \frac{1}{2c^2}\int d_3x\,\mu v^i v^i \int_0^p \frac{dp}{\sigma} +$$

$$+\frac{1}{2c^2}\int d_3x \int d_3x'\,\mu\mu' \frac{1}{|\underline{x}-\underline{x}'|}\left(\int_0^p \frac{dp}{\sigma} + \int_0^{p'} \frac{dp}{\sigma}\right) +$$

$$+\frac{k}{c^4}\int d_3x \int d_3x' \frac{1}{|\underline{x}-\underline{x}'|}(\mu' p + \mu p'),$$

where

$$\mu = \frac{k}{c^2}\sigma, \qquad \text{the functions } f' \text{ denote } f(x'),$$

$$[\underline{x},\underline{x}',\underline{x}''] = \frac{1}{|\underline{x}-\underline{x}'|}\frac{1}{|\underline{x}'-\underline{x}''|} + \frac{1}{|\underline{x}''-\underline{x}'|}\frac{1}{|\underline{x}''-\underline{x}|} + \frac{1}{|\underline{x}'-\underline{x}|}\frac{1}{|\underline{x}-\underline{x}''|}.$$

The functions μ and v^k have not been expanded, therefore the "Newtonian" part of (7) contains all the corrections up to the sixth order.

It is shown in [4] that the functional (7) is the correct Fokker-type Lagrangian for the perfect fluid, that means, it leads to the same equations of motion as those, which can be derived from (1). It is essential to note that only Newtonian pressures contribute to (7).

By a suitable procedure of variation the equations of motion of the fluid can be derived from (7). In the Newtonian approximation they reduce to

$$(8)\qquad \mu v^k{}_{,0} + \mu v^i v^k{}_{,i} = \mu \int d_3x'\,\mu' \left(\frac{1}{|\underline{x}-\underline{x}'|}\right)_{,k} - \frac{k}{c^4} p_{,k}.$$

3. Let us consider now the system of N rotating bodies of finite size, built from the perfect fluid. Our aim is to derive the Lagrange function for such a system. According to this problem we assume that the

functions μ, p and v^k in (7) vanish everywhere outside the volumes $\overset{A}{\Omega}$ $(A = 1, 2, ..., N)$ of the bodies. Inside the bodies we put

$$v^i(x) = \overset{A}{\xi}{}^i{}_{,0} + \overset{A}{\omega}{}^{ki}(x^k - \overset{A}{\xi}{}^k), \qquad x \in \overset{A}{\Omega} \tag{9}$$

where $\overset{A}{\xi}{}^i{}_{,0}$ denotes the velocity of the centre of mass of the A-th body

$$\overset{A}{\xi}{}^i = \frac{1}{\overset{A}{m}} \int_{\overset{A}{\Omega}} d_3 x \mu x^i, \qquad \overset{A}{m} = \int_{\overset{A}{\Omega}} d_3 x \mu(x); \tag{10}$$

$\overset{A}{\omega}{}^{ik} = -\overset{A}{\omega}{}^{ki}$ is the bivector of the angular velocity which depends on time only. We neglect, therefore, internal motions other than the rotation of the body as a whole. Furthermore, we assume that the dimensions (l) of the bodies are small when compared with the distances (L) between them, i.e., $l/L \ll 1$. The above assumptions will enable us to reduce the infinity of the degrees of freedom in (7) to a finite number of parameters describing the state of motion of the rotating bodies.

4. Before proceeding with calculation of the post-Newtonian Lagrangian it will be convenient to study some properties of the Newtonian motion. The Newtonian equations of motion of rotating bodies are easy to derive directly from (8) [1]. They read

$$\overset{A}{m}\,\overset{A}{\xi}{}^i{}_{,00} = \sum_{B \neq A} \overset{A}{m}\,\overset{B}{m} \left(\frac{1}{r_{AB}}\right)_{,\,\overset{A}{\xi}{}^i} + \ldots, \qquad r_{AB} = |\overset{A}{\underline{\xi}} - \overset{B}{\underline{\xi}}|; \tag{11}$$

$$\overset{A}{S}{}^{ik}{}_{,0} = \sum_{B \neq A} \overset{B}{m} \left[\left(\frac{1}{r_{AB}}\right)_{,\,\overset{A}{\xi}{}^k\,\overset{A}{\xi}{}^j} \overset{A}{I}{}^{ij} - \left(\frac{1}{r_{AB}}\right)_{,\,\overset{A}{\xi}{}^i\,\overset{A}{\xi}{}^j} \overset{A}{I}{}^{ik} \right] + \ldots, \quad A = 1 \ldots N, \tag{12}$$

where

$$\begin{cases} \overset{A}{m} = \int_{\overset{A}{\Omega}} d_3 x \mu(x), \\ \overset{A}{I}{}^i = \int_{\overset{A}{\Omega}} d_3 x \mu(x)(x^i - \overset{A}{\xi}{}^i) = 0, \\ \overset{A}{I}{}^{ik} = \int_{\overset{A}{\Omega}} d_3 x \mu(x)(x^i - \overset{A}{\xi}{}^i)(x^k - \overset{A}{\xi}{}^k), \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases} \tag{13}$$

constitute the set of the multipole momenta of the bodies, and $\overset{A}{S}{}^{ik} = \overset{A}{\omega}{}^{jk}\,\overset{A}{I}{}^{ji} - \overset{A}{\omega}{}^{ji}\,\overset{A}{I}{}^{jk}$ define the angular momentum of the A-th body. The terms omitted in (11) and (12) are of the higher order in l/L, they contain higher multipole momenta.

On the other hand, the Newtonian part of the Lagrangian (7) can be written in the form

$$\underset{\to 4}{\mathcal{L}} = \sum_{A=1}^{N} \left\{ -\overset{A}{m} + \frac{1}{2}\overset{A}{m}\,\overset{A}{\xi}{}^{i}{}_{,0}\,\overset{A}{\xi}{}^{i}{}_{,0} + \frac{1}{2}\,\overset{A}{\omega}{}^{ji}\,\overset{A}{\omega}{}^{jk}\,\overset{A}{I}{}^{ik} + \right.$$

$$\left. + \sum_{B \neq A} \left[\frac{\overset{A}{m}\,\overset{B}{m}}{r_{AB}} + \ldots, \right] + \overset{A}{\varepsilon} + \underset{\to 4}{\overset{A}{\mathcal{L}}}{}_{p} \right\}, \tag{14}$$

where

$$\overset{A}{\varepsilon} = \int\limits_{\overset{A}{\Omega}} d_3 x \int\limits_{\overset{A}{\Omega}} d_3 x' \, \mu\mu' \frac{1}{|\underline{x} - \underline{x}'|},$$

and $\underset{\to 4}{\overset{A}{\mathcal{L}}}{}_{p}$ denotes the pressure part of the Lagrangian of the A-th body.

Taking into account that

$$|\overset{A}{\underline{\xi}} - \overset{B}{\underline{\xi}}| \sim L, \quad |\underline{x} - \underline{x}'| \sim l, \quad \text{for} \quad x^i \in \overset{A}{\Omega} \quad \text{and} \quad x'^i \in \overset{A}{\Omega}$$

and making use of the virial theorem associated with the Eqs. (11), it can be shown that a consistent choice for the orders of the individual terms in (14) is

$$\overset{A}{m} \sim \left(\frac{l}{L}\right)^0, \quad \overset{A}{\xi}{}^{i}{}_{,0}\,\overset{A}{\xi}{}^{i}{}_{,0} \sim \left(\frac{l}{L}\right)^0, \quad \overset{A}{\omega}{}^2 l^2 \sim \left(\frac{l}{L}\right)^0, \quad \frac{\overset{A}{m}\,\overset{B}{m}}{r_{AB}} \sim \left(\frac{l}{L}\right)^0, \quad \overset{A}{\varepsilon} \sim \left(\frac{l}{L}\right)^{-1}. \tag{15}$$

The order of the pressure part remains undetermined for the present.

Inside the bodies the equations of the continuous matter are satisfied. For the A-th body we can write them in the form

$$\overset{A}{\xi}{}^{k}{}_{,0} + \overset{A}{\omega}{}^{jk}{}_{,0}(x^j - \overset{A}{\xi}{}^{j}) - \overset{A}{\omega}{}^{jk}\,\overset{A}{\omega}{}^{jl}(x^l - \overset{A}{\xi}{}^{l}) - \overset{A}{\varphi}{}_{,k} - \sum_{B \neq A} \overset{B}{\varphi}{}_{,k} = -\frac{1}{\mu}\frac{k}{c^4} p_{,k}, \tag{16}$$

where

$$\overset{A}{\varphi}(x) = \int\limits_{\overset{A}{\Omega}} d_3 x' \, \mu' \frac{1}{|\underline{x} - \underline{x}'|}.$$

The quantities

$$\overset{A}{\xi}{}^{k}{}_{,00} - \sum_{B \neq A} \overset{B}{\varphi}{}_{,k} \quad \text{and} \quad \overset{A}{\omega}{}^{jk}{}_{,0}(x^j - \overset{A}{\xi}{}^{j})$$

are of a higher order ($\sim (l/L)^2$) than the other terms in (16). Neglecting them, we obtain the equation

$$\frac{1}{\mu}\frac{k}{c^4} p_{,k} = \overset{A}{\varphi}{}_{,k} + \overset{A}{\omega}{}^{lj}\,\overset{A}{\omega}{}^{kj}(x^i - \overset{A}{\xi}{}^{i}), \qquad x^i \in \overset{A}{\Omega}. \tag{17}$$

The pressure part of the Lagrangian (14) can be determined by Eq. (17). Simple calculation gives

$$(18)\qquad \begin{cases} \dfrac{1}{c^2}\displaystyle\int_0^{p(x)} \frac{dp}{\sigma(p)} = \overset{A}{\varphi}(x) + \frac{1}{2}\,\overset{A}{\omega}{}^{ki}\,\overset{A}{\omega}{}^{kj}(x^i - \overset{A}{\xi}{}^{j})(x^i - \overset{A}{\xi}{}^{i}), \qquad x \in \overset{A}{\Omega} \\ \displaystyle\int_{\overset{A}{\Omega}} d_3 x\, p(x) = \frac{1}{6}\,\overset{A}{\varepsilon} - \frac{1}{3}\,\overset{A}{\omega}{}^{ji}\,\overset{A}{\omega}{}^{ki}\,\overset{A}{I}{}^{jk}, \end{cases}$$

It follows that $\mathscr{L}_p \sim (l/L)^{-1}$.

Moreover, the distribution of mass is, in a certain degree, determined by (17). Namely, the surfaces of equal density (equal pressure), called *equilibrium surfaces,* must satisfy the condition

$$(19)\qquad \int_A d_3 x'\, \mu(x') \frac{1}{|\underline{x} - \underline{x}'|} + \frac{1}{2}\,\overset{A}{\omega}{}^{ji}\,\overset{A}{\omega}{}^{jk}(x^i - \overset{A}{\xi}{}^{i})(x^k - \overset{A}{\xi}{}^{k}) = \text{const}$$

for x^i lying on the equilibrium surface.

It can be shown that Eq. (19) allows, in the used approximation, only the ellipsoidal form of the surfaces of equilibrium, i.e. the equation for such surfaces is

$$(20)\qquad r = a\left[1 + \alpha(a)\left(\frac{1}{3} - \cos^2\Theta\right)\right],$$

where r and Θ are the usual spherical co-ordinates, the centre of the system lies in the centre of the body, and $\Theta = 0$ on the axis of rotation of the body; the parameter a orders the equilibrium surfaces, in particular, for $a = a_m$, where a_m is the maximum value of a, Eq. (20) determines the shape of the external surface of the body. The function $\alpha(a)$, which determines the oblateness of the surfaces is proportional to the second power of the angular velocity and is of the first order in l L.

We see, that the problem of the Newtonian motion can be split by the l/L method of approximation into two problems: the internal problem dealing with the determination of the pressures on the basis of Eq. (17), and the external problem — the problem of motion in the strict sense. These two problems can be solved independently in the Newtonian approximation. In the post-Newtonian approximation the situation is different, but the solution of the internal problem in the Newtonian approximation is sufficient to solve the external post-Newtonian problem.

5. With the aid of the assumptions of sec. 3 and the results of sec. 4 we shall evaluate now the post-Newtonian Lagrangian for rotating bodies. We assume that the estimates (15) are valid in the post-Newtonian approximation, and we expand the Lagrangian with respect to l/L. Further, we eliminate the pressure terms by means of (18) and after a lengthy calculation *) we obtain

*) The calculations will be published in full in Acta Physica Polonica.

$$
\begin{aligned}
(21)\qquad \underset{(\to 6,\,\to l/L)}{\mathcal{L}} = \sum_{A=1}^{N} \Bigg\{ & -\overset{A}{m} + \frac{1}{2}\,\overset{A}{\xi}{}^{i}_{,0}\,\overset{A}{\xi}{}^{i}_{,0} + \frac{1}{2}\,\overset{A}{\omega}{}^{ji}\,\overset{A}{\omega}{}^{jk}\,\overset{A}{I}{}^{ik} + \\
& + \sum_{B\neq A} \frac{\overset{A}{m}\,\overset{B}{m}}{r_{AB}} + \frac{1}{2}\,\overset{A}{\xi}{}^{i}_{,0}\,\overset{A}{\xi}{}^{i}_{,0}\,\overset{A}{\omega}{}^{jk}\,\overset{A}{\omega}{}^{jl}\,\overset{A}{I}{}^{kl} + \frac{1}{2}\,\overset{A}{\xi}{}^{i}_{,0}\,\overset{A}{\xi}{}^{k}_{,0}\,\overset{A}{\omega}{}^{ji}\,\overset{A}{\omega}{}^{lk}\,\overset{A}{I}{}^{jl} + \\
& + \overset{A}{\xi}{}^{i}_{,0}\,\overset{A}{\omega}{}^{jl}\,\overset{A}{\omega}{}^{nl}\,\overset{A}{\omega}{}^{mi}\,\overset{A}{I}{}^{jmn} + \frac{3}{8}\,\overset{A}{\omega}{}^{ji}\,\overset{A}{\omega}{}^{ki}\,\overset{A}{\omega}{}^{nl}\,\overset{A}{\omega}{}^{ml}\,\overset{A}{I}{}^{jkmn} + \\
& + \sum_{B\neq A} \Bigg[\frac{2}{3}\,\frac{1}{r_{AB}}\,\overset{(A}{m}\,\overset{B)}{\omega}{}^{ji}\,\overset{B}{\omega}{}^{ki}\,\overset{B)}{I}{}^{jk} + \left(\frac{1}{r_{AB}}\right)_{,\overset{A}{\xi}{}^{i}} \Bigg(2\,\overset{[A}{m}\,\overset{A}{\xi}{}^{l}_{,0}\,\overset{B}{\omega}{}^{jl}\,\overset{B]}{I}{}^{ji} + \\
& + \frac{3}{2}\,\overset{[B}{m}\,\overset{A}{\xi}{}^{l}_{,0}\,\overset{A}{\omega}{}^{jl}\,\overset{A]}{I}{}^{kj} + \frac{3}{4}\,\overset{[B}{m}\,\overset{A}{\omega}{}^{jl}\,\overset{A}{\omega}{}^{ki}\,\overset{A]}{I}{}^{jki} \Bigg) \Bigg] + \\
& - \frac{1}{3}\,\overset{A}{\varepsilon} + \frac{1}{4}\,\overset{A}{\varepsilon}\,\overset{A}{\xi}{}^{i}_{,0}\,\overset{A}{\xi}{}^{i}_{,0} - \frac{1}{4}\,\overset{A}{B}{}^{ik}\,\overset{A}{\xi}{}^{i}_{,0}\,\overset{A}{\xi}{}^{k}_{,0} + \frac{1}{2}\,\overset{A}{\varepsilon}{}^{j}\,\overset{A}{\omega}{}^{ji}\,\overset{A}{\xi}{}^{i}_{,0} + \\
& - \frac{1}{2}\,\overset{A}{B}{}^{j,\,ik}\,\overset{A}{\xi}{}^{i}_{,0}\,\overset{A}{\omega}{}^{jk} + \frac{3}{4}\,\overset{A}{\varepsilon}{}^{ik}\,\overset{A}{\omega}{}^{ji}\,\overset{A}{\omega}{}^{jk} - \frac{1}{4}\,\overset{A}{B}{}^{jl,\,ik}\,\overset{A}{\omega}{}^{ji}\,\overset{A}{\omega}{}^{lk} + \\
& + \sum_{B\neq A} \Bigg[\frac{1}{6}\,\frac{1}{r_{AB}}\,\overset{(A}{m}\,\overset{B)}{\varepsilon} + \left(\frac{1}{r_{AB}}\right)_{,\overset{A}{\xi}{}^{i}} \left(\frac{1}{8}\,\overset{[B}{m}\,\overset{A]}{\varepsilon}{}^{i} + \frac{1}{8}\,\overset{[B}{m}\,\overset{A]}{B}{}^{j,\,ij} \right) \Bigg] \Bigg\}.
\end{aligned}
$$

The notation (A, B) for a symmetrical and $[A, B]$ — for an antisymmetrical combination of quantities is used. We have omitted in (21) the post-Newtonian terms which do not depend on rotation. They are identical with the known EIH terms for point-particles. The Lagrangian (21) depends, besides on the parameters of the motion $\overset{A}{\xi}{}^{i}$, $\overset{A}{\xi}{}^{i}_{,0}$, $\overset{A}{\omega}{}^{ik}$ and the multipole momenta of the bodies, on some quantities depending on the structure of the bodies, which are defined as follows:

$$
(22)\qquad \begin{cases}
\overset{A}{\varepsilon} = \int\limits_{\overset{A}{\Omega}} d_3 x\, \mu(x)\, \overset{A}{\varphi}(x), \\
\overset{A}{\varepsilon}{}^{i} = \int\limits_{\overset{A}{\Omega}} d_3 x\, \mu(x)\, (x^i - \overset{A}{\xi}{}^{i})\, \overset{A}{\varphi}(x), \\
\overset{A}{\varepsilon}{}^{ik} = \int\limits_{\overset{A}{\Omega}} d_3 x\, \mu(x)\, (x^i - \overset{A}{\xi}{}^{i})\, (x^k - \overset{A}{\xi}{}^{k})\, \overset{A}{\varphi}, \\
\overset{A}{B}{}^{ik} = \int\limits_{\overset{A}{\Omega}} d_3 x\, \mu(x)\, \overset{A}{b}{}^{ik}(x), \\
\overset{A}{B}{}^{j,\,ik} = \int\limits_{\overset{A}{\Omega}} d_3 x\, \mu(x)\, (x^j - \overset{A}{\xi}{}^{j})\, \overset{A}{b}{}^{ik}(x), \\
\overset{A}{B}{}^{jl,\,ik} = \int\limits_{\overset{A}{\Omega}} d_3 x\, \mu(x)\, (x^j - \overset{A}{\xi}{}^{j})\, (x^l - \overset{A}{\xi}{}^{l})\, \overset{A}{b}{}^{ik}(x),
\end{cases}
$$

where

$$\overset{A}{\varphi}(x) = \int\limits_{\overset{A}{\Omega}} d_3 x' \, \mu(x') \, \frac{1}{|\underline{x} - \underline{x}'|}; \quad \overset{A}{b}{}^{ik}(x) = \int\limits_{\overset{A}{\Omega}} d_3 x' \, \mu(x') \, \frac{(x^i - x'^i)(x^k - x'^k)}{|\underline{x} - \underline{x}|^3}.$$

The quantities (22) are of the fourth order in $1/c$ and of the order of $(l/L)^{-1}$ when compared to the analogous multipole momenta.

The Lagrangian (21) differs from the Lagrangian given by Fock [1] only by purely rotational terms.

The multipole momenta (13) and the quantities (22) depend on the distribution of mass inside the bodies, which, however, is not arbitrary in our model. Making use of the form (20) of the surfaces of equal density we can write

$$(23)\quad \begin{cases} \overset{A}{I}{}^{ik} = \delta_{ik} \underset{(0)}{\overset{A}{I}} + \underset{(1)}{\overset{A}{I}}{}^{ik} + \ldots, \\ \overset{A}{I}{}^{ijk} = 0 + \underset{(1)}{\overset{A}{I}}{}^{ijk} + \ldots, \\ \overset{A}{\varepsilon} = \underset{(-1)}{\overset{A}{\varepsilon}}, \\ \overset{A}{\varepsilon}{}^{i} = \underset{(-1)}{\overset{A}{\varepsilon}}{}^{i} + \underset{(0)}{\varepsilon}{}^{i} + \ldots, \quad \underset{(-1)}{\varepsilon}{}^{i} = 0, \\ \overset{A}{\varepsilon}{}^{ik} = \underset{(-1)}{\overset{A}{E}} \delta_{ik} + \underset{(0)}{\overset{A}{\varepsilon}}{}^{ik} + \ldots, \\ \overset{A}{B}{}^{ik} = \frac{1}{3} \delta_{ik} \underset{(-1)}{\overset{A}{\varepsilon}} + \underset{(0)}{\overset{A}{B}}{}^{ik} + \underset{(1)}{\overset{A}{B}}{}^{ik} + \ldots, \quad \underset{(0)}{\overset{A}{B}}{}^{ik} = \left(\frac{2}{3} \delta_{ik} \overset{A}{\omega}{}^{jl} \overset{A}{\omega}{}^{jl} - 2 \overset{A}{\omega}{}^{ji} \overset{A}{\omega}{}^{jk} \right) \underset{(0)}{\overset{A}{B}}, \\ \overset{A}{B}{}^{j,ik} = \frac{1}{3} \delta_{ik} \underset{(-1)}{\overset{A}{\varepsilon}}{}^{j} + \underset{(0)}{\overset{A}{B}}{}^{j,ik} + \ldots, \\ \overset{A}{B}{}^{jl,ik} = \frac{1}{3} \delta^{jl} \underset{(-1)}{\overset{A}{\varepsilon}}{}^{ik} + \underset{(0)}{\overset{A}{B}}{}^{jl,ik} + \ldots, \end{cases}$$

where the indices below indicate the order of the corresponding quantities in l/L. The leading terms in (23) correspond to spherically symmetric bodies, further terms are due to the oblateness of the bodies.

Substituting (23) into (21) and neglecting, as usual, the terms of an order higher than $(l/L)^1$, we obtain

$$(24)\quad \mathcal{L} = \sum_{A=1}^{N} \left\{ -\overset{A}{m} + \frac{1}{2} \overset{A}{m} \overset{A}{\xi}{}^{i}_{,0} \overset{A}{\xi}{}^{i}_{,0} + \frac{1}{2} \overset{A}{\omega}{}^{ik} \overset{A}{\omega}{}^{ik} \overset{A}{I} + \sum_{B \neq A} \frac{1}{2} \frac{\overset{A}{m} \overset{B}{m}}{r_{AB}} + \right.$$

$$+ \frac{1}{2} \overset{A}{\xi}{}^{i}_{,0} \overset{A}{\xi}{}^{i}_{,0} \overset{A}{\omega}{}^{kl} \overset{A}{\omega}{}^{kl} \left(\overset{A}{I} - \frac{1}{3} \overset{A}{B} \right) + \frac{1}{2} \overset{A}{\xi}{}^{i}_{,0} \overset{A}{\xi}{}^{k}_{,0} \overset{A}{\omega}{}^{ji} \overset{A}{\omega}{}^{jk} \left(\overset{A}{I} + \overset{A}{B} \right) +$$

$$\left. + \sum_{B \neq A} \left[\frac{2}{3} \frac{1}{r_{AB}} \overset{(A}{m} \overset{B}{\omega}{}^{ji} \overset{B}{\omega}{}^{ji} \overset{B)}{I} + \left(\frac{1}{r_{AB}} \right)_{,\overset{A}{\xi}{}^{i}} \left(2 \overset{[B}{m} \overset{B}{\xi}{}^{l}_{,0} \overset{A}{\omega}{}^{il} \overset{A]}{I} + \right. \right. \right.$$

$$+\frac{3}{2}\overset{[B}{m}\overset{A}{\xi^l}_{,0}\overset{A}{\omega^{il}}\overset{A]}{I}\Big)\Big]-\frac{1}{3}\overset{A}{\varepsilon}+\frac{1}{6}\overset{A}{\varepsilon}\overset{A}{\xi^i}_{,0}\overset{A}{\xi^i}_{,0}+\frac{1}{2}\overset{A}{\varepsilon^j}\overset{A}{\omega^{ji}}\overset{A}{\xi^i}_{,0}+$$

$$+\sum_{B\neq A}\left[\frac{1}{r_{AB}}\frac{1}{6}\overset{(A}{m}\overset{B)}{\varepsilon}+\left(\frac{1}{r_{AB}}\right)_{,\overset{A}{\xi^i}}\frac{1}{6}\overset{[B}{m}\overset{A]}{\varepsilon^i}\right]\Big\}.$$

We have omitted in (24), for simplicity, the terms proportional to higher powers of the angular velocity.

Now, we shall further simplify the Langrangian (24). Let us introduce a new density

$$\mu_R=\mu\left(1+\frac{1}{3}\overset{A}{\varphi}\right)\quad\text{for}\quad x\in\overset{A}{\Omega}. \tag{25}$$

The corresponding quantities $\overset{A}{m}_R$, $\overset{A}{\xi^i_R}$, $\overset{A}{\xi^i_{R,0}}$, $\overset{A}{I^{ik}_R}$ will be:

$$\begin{cases}\overset{A}{m}=\overset{A}{m}+\frac{1}{3}\overset{A}{\varepsilon}\,,\\ \overset{A}{\xi^i_R}=\overset{A}{\xi^i}+\frac{1}{3}\frac{1}{\overset{A}{m}}\overset{A}{\varepsilon^i}\,,\\ \overset{A}{\xi^i_{R,0}}=\overset{A}{\xi^i}_{,0}+\frac{1}{3}\frac{1}{\overset{A}{m}}\overset{A}{\omega^{ji}}\overset{A}{\varepsilon^j}\,,\\ \overset{A}{I^{ik}}=\overset{A}{I^{ik}_R}+\frac{1}{3}\overset{A}{\varepsilon^{ik}}\,.\end{cases} \tag{26}$$

Introducing them into the Lagrangian, we obtain

$$\mathcal{L}=\sum_{A=1}^{N}\Big\{-\overset{A}{m}_R+\frac{1}{2}\overset{A}{m}_R\overset{A}{\xi^i_{R,0}}\overset{A}{\xi^i_{R,0}}+\frac{1}{2}\overset{A}{\omega^{ji}}\overset{A}{\omega^{ji}}\overset{A}{I}_R+ \tag{27}$$

$$+\frac{1}{2}\overset{A}{\xi^i_{R,0}}\overset{A}{\xi^i_{R,0}}\overset{A}{\omega^{kl}}\overset{A}{\omega^{kl}}\left(\overset{A}{I}_R-\frac{1}{3}\overset{A}{B}\right)+\frac{1}{2}\overset{A}{\xi^i_{R,0}}\overset{A}{\xi^k_{R,0}}\overset{A}{\omega^{ji}}\overset{A}{\omega^{jk}}(\overset{A}{I}_R+\overset{A}{B})+$$

$$+\sum_{B\neq A}\left[\frac{2}{3}\frac{1}{r_{AB}}\overset{(A}{m}_R\overset{B}{\omega^{ji}}\overset{B}{\omega^{ji}}\overset{B)}{I}+2\left(\frac{1}{r_{AB}}\right)_{,\overset{A}{\xi^i}}\Big(\overset{[B}{m}_R\overset{B}{\xi^k_{R,0}}\overset{A}{\omega^{ik}}\overset{A]}{I}_R+\right.$$

$$\left.+\frac{3}{4}\overset{[B}{m}_R\overset{A}{\xi^k_{R,0}}\overset{A}{\omega^{ik}}\overset{A]}{I}_R\Big)\right]\Big\}.$$

The above procedure consists in a renormalisation of the density of mass. The density μ is not a directly observable quantity. Information about it can be obtained only by comparing the consequences of the equations of motion with observations. Thus, the new definition of μ does not change the physical contents of the equations of motion.

The formula (11) gives the final form of our Lagrangian. Beyond the multipole momenta $\overset{A}{m}$, $\overset{A}{I}$ the Lagrangian depends on some structural constants $\overset{A}{B}$. If we performed the calculations to a higher order in l/L,

more similar constants would appear. Their appearance in the Lagrangian is connected with the non-linear character of the theory of gravitation and indicates that a gravitational "skeleton" consisting of multipole momenta does not exist in our model of bodies.

To compare (27) with Tulczyjew's results [2] we must remember that his paper deals with spherically symmetric bodies. In our model rotations do not allow spherical symmetry. Departures from the spherical symmetry are, however, proportional to ω^2. Omitting all such terms in the Lagrangian and accepting Tulczyjew's definition of the centre of mass we obtain a Lagrangian from which Tulczyjew's equations of motion follow *).

The Lagrangian (27) is also in agreement with the Lagrangian for spherically symmetric bodies which is given by Kalitzin [6] on the basis of Fock's theory.

The equations of the translatory and rotational motion will be derived from (27) in a subsequent paper.

I wish to express my thanks to Dr J. Plebański for formulating the problem and many valuable discussions. I am greatly indebted to Professor L. Infeld for his kind interest in this work. I am grateful to Dr A. Trautman, Dr W. Tulczyjew and Dr S. Bażański for numerous discussions.

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*) One must take into account some slight mistakes in Tulczyjew's paper.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VIII, No. 4, 1960

The Equations of Motion of Rotating Oblate Bodies in the General Theory of Relativity

by

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Presented by L. INFELD on February 25, 1960

In a previous paper [1] we have found the post-Newtonian Lagrangian for the system of rotating bodies. The object of the present paper is to derive from this Lagrangian the equations of the translatory and rotational motion of two bodies and to compute the secular corrections for the Newtonian motion due to rotations.

1. The Lagrangian for the system of two rotating bodies up to terms of the order l/L *), as given in [1], is

$$
\begin{aligned}
(1)\quad \underset{\to 6}{\mathcal{L}} = \sum_{A=1}^{2} \Bigg\{ & \frac{1}{2} \overset{A}{m} \overset{A}{\xi}{}^{i}{}_{,0} \overset{A}{\xi}{}^{i}{}_{,0} + \frac{1}{2} \overset{A}{\omega}{}^{ik} \overset{A}{\omega}{}^{ik} \overset{A}{I} + \sum_{B \neq A} \frac{1}{2} \overset{A}{m} \overset{B}{m} \frac{1}{r_{AB}} + \\
& + \overset{A}{\xi}{}^{i}{}_{,0} \overset{A}{\xi}{}^{i}{}_{,0} \overset{A}{\omega}{}^{jk} \overset{A}{\omega}{}^{jk} \left(\overset{A}{I} - \frac{1}{3} \overset{A}{B} \right) + \frac{1}{2} \overset{A}{\xi}{}^{i}{}_{,0} \overset{A}{\xi}{}^{k}{}_{,0} \overset{A}{\omega}{}^{ji} \overset{A}{\omega}{}^{jk} (\overset{A}{I} + \overset{A}{B}) + \\
& + \sum_{B \neq A} \Bigg[\frac{2}{3} \frac{1}{r_{AB}} (\overset{A}{m} \overset{B}{\omega}{}^{ik} \overset{B}{\omega}{}^{ik} \overset{B}{I} + \overset{B}{m} \overset{A}{\omega}{}^{ik} \overset{A}{\omega}{}^{ik} \overset{A}{I}) + \\
& + 2 \left(\frac{1}{r_{AB}} \right)_{,\overset{A}{\xi}{}^{i}} \left(\overset{B}{m} \overset{B}{\xi}{}^{k}{}_{,0} \overset{A}{\omega}{}^{ik} \overset{A}{I} + \frac{3}{4} \overset{B}{m} \overset{A}{\xi}{}^{k}{}_{,0} \overset{A}{\omega}{}^{jk} \overset{A}{I} - \overset{A}{m} \overset{A}{\xi}{}^{k}{}_{,0} \overset{B}{\omega}{}^{ik} \overset{B}{I} - \frac{3}{4} \overset{A}{m} \overset{B}{\xi}{}^{k}{}_{,0} \overset{B}{\omega}{}^{ik} \overset{B}{I} \right) \Bigg] \Bigg\},
\end{aligned}
$$

where $\overset{A}{\xi}{}^{i}$ are the co-ordinates of the centres of mass of the bodies, $r_{AB} = |\overset{A}{\xi} - \overset{B}{\xi}|$, $\overset{A}{\omega}{}^{ik} = -\overset{A}{\omega}{}^{ki}$ denote their angular velocities, $\overset{A}{m}$ — the masses of the bodies, $\overset{A}{I}$ — the diagonal parts of the quadrupole momenta ($\overset{A}{I}{}^{ik} = \delta_{ik} \overset{A}{I}$) and $\overset{A}{B}$ are constant quantities describing the oblateness of the bodies. The EIH terms are neglected in (1).

*) l — is a length characterizing the dimensions of the bodies and L the distance between them.

Introducing the following notations:

$$(2)\qquad \Xi^i = \frac{1}{M}(\overset{1}{m}\,\overset{1}{\xi}{}^i + \overset{2}{m}\,\overset{2}{\xi}{}^i), \qquad \xi^i = \overset{1}{\xi}{}^i - \overset{2}{\xi}{}^i, \qquad \dot{\xi}^i = \xi^i{}_{,0}, \quad r = (\xi^i\,\xi^i)^{1/2},$$

$$M = \overset{1}{m} + \overset{2}{m}, \qquad \mathfrak{M} = \overset{1}{m}\,\overset{2}{m}/M$$

and restricting ourselves to the centre of the inertia system (total momentum $= 0$), we can write the Lagrangian (1) in the form

$$(3)\qquad \underset{\to 6}{\mathcal{L}} = \frac{1}{2}\mathfrak{M}\,\dot{\xi}^i\,\dot{\xi}^i + \mathfrak{M}\,M\frac{1}{r} + \frac{1}{2}\,\overset{1}{\omega}{}^{ik}\,\overset{1}{\omega}{}^{ik}\,\overset{1}{I} + \frac{1}{2}\,\overset{2}{\omega}{}^{ik}\,\overset{2}{\omega}{}^{ik}\,\overset{2}{I} +$$

$$+ \frac{1}{2}\,\dot{\xi}^i\,\dot{\xi}^i\left[\left(\frac{\overset{1}{m}}{M}\right)^2 \overset{1}{\omega}{}^{lk}\,\overset{1}{\omega}{}^{lk}\left(\overset{1}{I} - \frac{1}{3}\overset{1}{B}\right) + \left(\frac{\overset{1}{m}}{M}\right)^2 \overset{2}{\omega}{}^{lk}\,\overset{2}{\omega}{}^{lk}\left(\overset{2}{I} - \frac{1}{3}\overset{2}{B}\right)\right] +$$

$$+ \frac{1}{2}\,\dot{\xi}^i\,\dot{\xi}^k\left[\left(\frac{\overset{2}{m}}{M}\right)^2 \overset{1}{\omega}{}^{ji}\,\overset{1}{\omega}{}^{jk}\,(\overset{1}{I} + \overset{1}{B}) + \left(\frac{\overset{1}{m}}{M}\right)^2 \overset{2}{\omega}{}^{ji}\,\overset{2}{\omega}{}^{jk}\,(\overset{2}{I} + \overset{2}{B})\right] +$$

$$+ \frac{4}{3}\,\frac{1}{r}\,(\overset{1}{m}\,\overset{2}{\omega}{}^{ik}\,\overset{2}{\omega}{}^{ik}\,\overset{2}{I} + \overset{2}{m}\,\overset{1}{\omega}{}^{ik}\,\overset{1}{\omega}{}^{ik}\,\overset{1}{I}) + 4\,\mathfrak{M}\left(\frac{1}{r}\right)_{,j}\left[\left(1 + \frac{3}{4}\,\frac{\overset{2}{m}}{\overset{1}{m}}\right)\overset{1}{\omega}{}^{ji}\,\overset{1}{I} +\right.$$

$$\left. + \left(1 + \frac{3}{4}\,\frac{\overset{1}{m}}{\overset{2}{m}}\right)\overset{2}{\omega}{}^{ji}\,\overset{2}{I}\right]\dot{\xi}^i.$$

The velocity of the centre of mass $\dot{\Xi}^i$ has been neglected for the same reason as done by Fock [2] in the two bodies problem without rotation.

From the action principle

$$(4)\qquad \delta\int dx^0\,\underset{\to 6}{\mathcal{L}} = 0$$

we obtain, by carrying out the variation with respect to ξ^i, the equations of the translatory motion

$$(5)\qquad \mathfrak{M}\,\ddot{\xi}^i - \mathfrak{M}\,M\left(\frac{1}{r}\right)_{,i} = \frac{4}{3}\left(\frac{1}{r}\right)_{,i}(\overset{1}{m}\,\overset{2}{\omega}{}^{jk}\,\overset{2}{\omega}{}^{jk}\,\overset{2}{I} + \overset{2}{m}\,\overset{1}{\omega}{}^{jk}\,\overset{1}{\omega}{}^{jk}\,\overset{1}{I}) +$$

$$- M\left(\frac{1}{r}\right)_{,i}\left[\left(\frac{\overset{2}{m}}{M}\right)^2 \overset{1}{\omega}{}^{jk}\,\overset{1}{\omega}{}^{jk}\left(\overset{1}{I} - \frac{1}{3}\overset{1}{B}\right) + \left(\frac{\overset{1}{m}}{M}\right)^2 \overset{2}{\omega}{}^{jk}\,\overset{2}{\omega}{}^{jk}\left(\overset{2}{I} - \frac{1}{3}\overset{2}{B}\right)\right] +$$

$$- M\left(\frac{1}{r}\right)_{,k}\left[\left(\frac{\overset{2}{m}}{M}\right)^2 \overset{1}{\omega}{}^{ji}\,\overset{1}{\omega}{}^{jk}(\overset{1}{I} + \overset{1}{B}) + \left(\frac{\overset{1}{m}}{M}\right)^2 \overset{2}{\omega}{}^{ji}\,\overset{2}{\omega}{}^{jk}\,(\overset{2}{I} + \overset{2}{B})\right] +$$

$$+ 4\,\mathfrak{M}\left(\frac{1}{r}\right)_{,ki}\left[\left(1 + \frac{3}{4}\,\frac{\overset{2}{m}}{\overset{1}{m}}\right)\overset{1}{\omega}{}^{kj}\,\overset{1}{I} + \left(1 + \frac{3}{4}\,\frac{\overset{1}{m}}{\overset{2}{m}}\right)\overset{2}{\omega}{}^{kj}\,\overset{2}{I}\right]\dot{\xi}^j +$$

$$+ 4\,\mathfrak{M}\left(\frac{1}{r}\right)_{,kj}\left[\left(1 + \frac{3}{4}\,\frac{\overset{2}{m}}{\overset{1}{m}}\right)\overset{1}{\omega}{}^{ki}\,\overset{1}{I} + \left(1 + \frac{3}{4}\,\frac{\overset{1}{m}}{\overset{2}{m}}\right)\overset{2}{\omega}{}^{ki}\,\overset{2}{I}\right]\dot{\xi}^j.$$

In order to derive the equations of the rotational motion it will be convenient to introduce new rotational variables $\overset{A}{a}{}^{ik}$. They are defined as follows. If x^k denote the co-ordinates of the point x'^k of the body with respect to a system in which the body is at rest, then

$$x^i - \overset{A}{\xi}{}^i = \overset{A}{a}{}^{ij}(x'^j - \overset{A}{\xi}{}^j), \qquad A = 1, 2 \tag{6}$$

and the orthogonality conditions

$$\overset{A}{a}{}^{ji}\,\overset{A}{a}{}^{jk} = \delta_{ik} \tag{7}$$

are satisfied. The coefficients $\overset{A}{a}{}^{ik}$ give full information about the state of rotation of the bodies.

Making use of the equality

$$\overset{A}{\omega}{}^{ik} = \overset{A}{\omega}{}^{ji}\,\overset{A}{\dot{a}}{}^{jk}, \tag{8}$$

which follows from (6), we express the Lagrangian (3) in terms of a^{ik} and $\dot{a}^{ik}$. The action principle with respect to the variables a^{ik}, together with the subsidiary conditions (7) leads to the equations of the rotational motion

$$\begin{aligned}\frac{d}{dx^0}\left[\overset{1}{S}{}^i + \frac{\overset{2}{m}}{M}\left(\frac{4}{3} + \frac{\overset{2}{m}}{M}\left(\frac{3}{2} + \frac{1}{6}\,\frac{\overset{1}{B}}{\overset{1}{I}}\right)\right)\dot{\xi}^k\,\dot{\xi}^k\,\overset{1}{S}{}^i - \frac{1}{2}\,\frac{\overset{2}{m}{}^2}{M^2}\,\dot{\xi}^i\,\dot{\xi}^k\left(1 + \frac{\overset{1}{B}}{\overset{1}{I}}\right)\overset{1}{S}{}^k\right] = \\ = -\frac{\overset{2}{m}{}^2}{M^2}\,\varepsilon^{ink}\,\dot{\xi}^l\,\dot{\xi}^k\,\overset{1}{\omega}{}^n\,\overset{1}{\omega}{}^l\,(\overset{1}{I} + \overset{1}{B}) + 2\left(\frac{1}{r}\right)_{,i}\left(1 + \frac{3}{4}\,\frac{\overset{2}{m}}{\overset{1}{m}}\right)\mathfrak{M}\,\dot{\xi}^k\,\overset{1}{S}{}^k + \\ - 2\,\mathfrak{M}\left(\frac{1}{r}\right)_{,k}\left(1 + \frac{3}{4}\,\frac{\overset{2}{m}}{\overset{1}{m}}\right)\dot{\xi}^i\,\overset{1}{S}{}^k,\end{aligned} \tag{9}$$

and similar equations for the second body.

We have introduced in (9) a new notation

$$\overset{A}{\omega}{}^i = \frac{1}{2}\,\varepsilon^{ijk}\,\overset{A}{\omega}{}^{jk}, \qquad \overset{A}{S}{}^i = \frac{1}{2}\,\varepsilon^{ijk}\,\overset{A}{S}{}^{jk}, \qquad A = 1, 2,$$

where ε^{ijk} is the permutation symbol.

2. It is known that, when the equations of motion follow from an action principle, certain conservation laws are attached to the invariance of the Lagrangian with regard to infinitesimal transformations. Some difficulties arise, however, in the case of rotating bodies, due to the fact that $\overset{A}{a}{}^{ik}$ are not independent variables. The formulae for the conserved mechanical quantities, as given, e.g., by Hill [3], cannot be used here. We shall derive the conservation law of the angular momentum by making use of the explicit form of the equations of motion.

From the invariance of the Lagrangian (3) with respect to infinitesimal rotations:

$$\xi^i \to \xi^i + \varepsilon^{ijk}\,\xi^k\,\delta\vartheta^j, \qquad \omega^i \to \omega^i + \varepsilon^{ijk}\,\omega^k\,\delta\vartheta^j$$

we derive the equation

$$\frac{d}{dx^0}L^k = \varepsilon^{kji}\,\dot{\xi}^i\,\xi^m O^{mj} + 2\,\mathfrak{M}\,M\left[\left(\frac{1}{r}\right)_{,i} S^i\,\dot{\xi}^k - \left(\frac{1}{r}\right)_{,k} S^i\,\dot{\xi}^i\right], \tag{10}$$

where

$$L^k = \varepsilon^{ikj}\,\frac{\partial \mathcal{L}}{\partial \xi^i_{,0}}\,\xi^j$$

is the orbital angular momentum of the system,

$$MS^k = \left(1 + \frac{3}{4}\,\frac{\overset{2}{m}}{\overset{1}{m}}\right)\overset{1}{S}{}^k + \left(1 + \frac{3}{4}\,\frac{\overset{1}{m}}{\overset{2}{m}}\right)\overset{2}{S}{}^k, \tag{11}$$

$$O^{ik} = \left(\frac{\overset{2}{m}}{M}\right)^2 (I + \overset{1}{B})\,\overset{1}{\omega}{}^i\,\overset{1}{\omega}{}^k + \left(\frac{\overset{1}{m}}{M}\right)^2 (I + \overset{2}{B})\,\overset{2}{\omega}{}^i\,\overset{2}{\omega}{}^k. \tag{12}$$

Eq. (10) shows that the orbital angular momentum, in general, is not conserved.

On the other hand, Eqs. (9) give

$$\frac{d}{dx^0}\left[\overset{1}{S}{}^k + \overset{2}{S}{}^k + \dot{\xi}^i\,\dot{\xi}^i\,\widetilde{S}^k + \dot{\xi}^i\,\dot{\xi}^k\,\widetilde{\widetilde{S}}{}^i\right] = \tag{13}$$

$$= \varepsilon^{kij}\,\dot{\xi}^i\,\xi^m O^{mj} + 2\,\mathfrak{M}\,M\left[\left(\frac{1}{r}\right)_{,k} S^i\,\dot{\xi}^i - \left(\frac{1}{r}\right)_{,i} S^i\,\dot{\xi}^k\right],$$

where

$$\begin{cases} \widetilde{S}^i = \dfrac{\overset{2}{m}}{M}\left[\dfrac{4}{3} + \dfrac{\overset{2}{m}}{M}\left(\dfrac{3}{2} + \dfrac{1}{6}\,\overset{1}{b}\right)\right]\overset{1}{S}{}^i + \dfrac{\overset{1}{m}}{M}\left[\dfrac{4}{3} + \dfrac{\overset{1}{m}}{M}\left(\dfrac{3}{2} + \dfrac{1}{6}\,\overset{2}{b}\right)\right]\overset{2}{S}{}^i, \\ \widetilde{\widetilde{S}}{}^i = -\dfrac{1}{2}\left[\dfrac{\overset{2}{m}{}^2}{M^2}(1 + \overset{1}{b})\,\overset{1}{S}{}^i + \dfrac{\overset{1}{m}{}^2}{M^2}(1 + \overset{2}{b})\,\overset{2}{S}{}^i\right], \qquad \overset{A}{b} = \overset{A}{B}\,\overset{A}{I}, \qquad A = 1, 2. \end{cases} \tag{14}$$

By comparing (10) and (13) we obtain the law of conservation of the total angular momentum

$$\frac{d}{dx^0}J^k = 0, \tag{15}$$

$$J^k = L^k + \overset{1}{S}{}^k + \overset{2}{S}{}^k + \dot{\xi}^i\,\dot{\xi}^i\,\widetilde{S}^k + \dot{\xi}^i\,\dot{\xi}^k\,\widetilde{\widetilde{S}}{}^i. \tag{16}$$

Eq. (16) constitutes a generalization of the analogous expression for the spherically symmetric bodies [4].

3. We compute now the secular corrections for the Newtonian motion. We can write Eqs. (5) in the form

$$\mathfrak{M}\,\ddot{\xi}^i - \mathfrak{M}\,M\left(\frac{1}{r}\right)_{,i} = F^i_{(1)} + F^i_{(2)}, \tag{17}$$

where the disturbing forces F^i are:

$$(18)\qquad F^i_{(1)} = -6M\mathfrak{M}\frac{1}{r^5}\left|\varepsilon^{jkn}\dot{\xi}^j\xi^i\xi^n S^k + \varepsilon^{ink}\dot{\xi}^j\xi^j\xi^n S^k\right| + 4\,\mathfrak{M}M\frac{1}{r^3}\varepsilon^{ijk}S^j\dot{\xi}^k,$$

$$(19)\qquad F^i_{(2)} = -O^{ik}\frac{x^k}{r^3}.$$

The terms proportional to $(1/r)_{,i}$ which do not contribute to the secular corrections, are neglected in (17). We note that $F^i_{(1)}$ is linear in the angular velocities, and $F^i_{(2)}$ depends on squares of ω^k.

Let the Newtonian motion be bounded. We assume, following the Lagrange theory of osculating orbits, that the disturbing forces produce slow motions of the Newtonian ellipse. Let e and p be the eccentricity

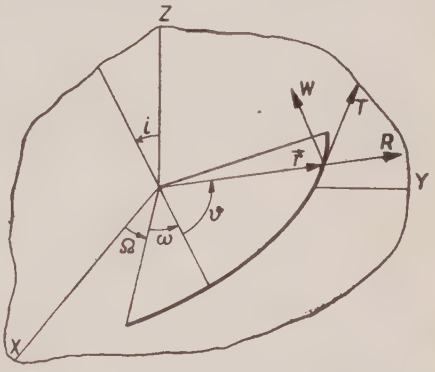

Fig. 1. The orientation of the orbit in space. Ω determines the position of the line of nodes, ω — the angular distance from the line of nodes to the periastron i — the inclination of the orbit to xy-plane, r is the current radius vector of the body

and the semi-latus-rectum of the ellipse. Let ω, Ω, i determine the orientation of the ellipse in the space (cf. Fig. 1). The variation of the parameters e, p, ω, Ω, i is given (cf. [5]) by the system of equations:

$$(20)\qquad \begin{cases} \dfrac{dp}{d\vartheta} = 2M^{-3/2}r^3T; \\ \dfrac{de}{d\vartheta} = M^{-3/2}r^2\left[R\sin\vartheta + \left(1+\dfrac{r}{p}\right)T\sin\vartheta - \dfrac{r}{p}\,ctg\,i\sin uW\right]; \\ \dfrac{di}{d\vartheta} = M^{-3/2}r^2p^{-1}\cos uW, \\ \dfrac{d\Omega}{d\vartheta} = M^{-3/2}r^3p^{-1}\sin u\,(\sin i)^{-1}W; \\ \dfrac{d\omega}{d\vartheta} = M^{-3/2}r^2\left[-e^{-1}R\cos\vartheta + e^{-1}\left(1+\dfrac{r}{p}\right)T\sin\vartheta - \dfrac{r}{p}\,ctgi\sin uW\right], \end{cases}$$

where ϑ is the current angle,

$$\int_{\vartheta_0}^{\vartheta} d\vartheta\,(1+e\cos\vartheta^{-1}) = (M/p^3)^{1/2}(t-t_0),\quad u=\omega+\vartheta,$$

R is the component of the disturbing force along the current radius vector $\vec{r}$, T — the component tangential to the projected orbit, W — the

component orthogonal to the orbital plane, so, that RTW form a right-handed triad.

We calculate first the secular corrections due to the force (18). We choose the z-axis in the direction of the angular momentum S^k and after simple calculation we obtain

$$(21)\qquad \begin{cases} \Delta^{(1)}\Omega = \displaystyle\int_0^{2\pi} d\vartheta \frac{d\Omega}{d\vartheta} = \frac{4\pi M}{p}\frac{|\vec{S}|}{\sqrt{pM}}, \qquad \Delta^{(1)}\omega = -\frac{12\pi M}{p}\frac{|\vec{S}|}{\sqrt{pM}}\cos i \\ \Delta^{(1)} e = \Delta^{(1)} p = \Delta^{(1)} i = 0. \end{cases}$$

The secular motion, due to the forces linear in ω^k, consists therefore, of the rotation $(\Delta^{(1)}\omega)$ of the Newtonian ellipse in a plane, which, at the same time, performs a precessional motion $(\Delta^{(1)}\Omega)$ about the constant vector $\vec{S}$.

In the particular case, when the angular momentum $\vec{S}$ is perpendicular to the plane of the Newtonian motion, i.e. $i = 0$, the orbit lies in a plane and Eqs. (21) reduce to

$$(22)\qquad \Delta^{(1)}\omega = -8\pi\frac{M}{p}\frac{|\vec{S}|}{\sqrt{pM}}.$$

This case was investigated by Tulczyjew [6] and Kalitzin [7]; the same value for the periastron advance was obtained.

Introducing into (20) the disturbing force (19) and assuming for the sake of simplicity, that

$$\overset{1}{m} = \overset{2}{m} = \frac{M}{2}, \quad \overset{1}{I} = \overset{2}{I} = I, \quad \overset{1}{B} = \overset{2}{B} = B, \quad \overset{1}{\vec{\omega}} \perp \overset{2}{\vec{\omega}}, \quad |\overset{1}{\vec{\omega}}| = |\overset{2}{\vec{\omega}}| = |\vec{\omega}|,$$

we obtain

$$(23)\qquad \begin{cases} \Delta^{(2)} p = -\frac{1}{2}\pi\frac{I+B}{M}|\vec{\omega}|^2 p e^2 [\sin 2\omega\cos^2\Omega + \cos 2\omega \sin 2\omega \cos i - \\ \qquad - \sin 2\omega \sin^2\Omega \cos^2 i]; \\ \Delta^{(2)} e = \frac{1}{2}\frac{e}{p}\Delta^{(2)} p; \\ \Delta^{(2)}\omega = \pi\frac{I+B}{M}|\vec{\omega}|^2\left[\sin^2\Omega\cos^2 i + \frac{1}{4}e^2(1+2\sin^2\omega)\sin^2\Omega\cos^2 i - \right. \\ \qquad \left. - \frac{1}{8}e^2 \sin 2\omega \sin 2\Omega\right]; \\ \Delta^{(2)}\Omega = \pi\frac{I+B}{M}|\vec{\omega}|^2\left[\sin^2\Omega\cos i + \frac{1}{4}e^2(1+2\sin^2\omega)\sin^2\Omega\cos i - \right. \\ \qquad \left. - \frac{1}{8}e^2 \sin 2\omega \sin 2\Omega\right]; \\ \Delta^{(2)} i = -\frac{1}{2}\pi\frac{I+B}{M}|\vec{\omega}|^2\left[\sin 2\Omega \sin i + \frac{1}{4}e^2(1+2\cos^2\omega)\sin 2\Omega \sin i - \right. \\ \qquad \left. - \frac{1}{4}\sin 2\omega \sin^2\Omega \sin 2i\right]. \end{cases}$$

The angle Ω in (23) determines the angular distance from the line of nodes to the yz-plane, in which $\overset{1}{\vec{\omega}}$ and $\overset{2}{\vec{\omega}}$ lie. The secular corrections Δe and Δp are very small ($\sim e^2$); it can be shown that they are periodic functions of time.

When the orbits are nearly circular $e \approx 0$, we have:

$$\text{(24)} \qquad \begin{cases} \Delta^{(2)} e = \Delta^{(2)} p = 0\,; \\ \Delta^{(2)} \Omega = \dfrac{I+B}{M} |\vec{\omega}|^2 \sin^2 \Omega (\cos^2 i - \cos i)\,; \\ \Delta^{(2)} i = \dfrac{1}{2} \pi \dfrac{I+B}{M} |\vec{\omega}|^2 \sin 2\Omega \sin i\,. \end{cases}$$

By means of the method of light rays (cf. [8]) we may verify that the values of $\Delta\Omega$ and Δi observed at spatial infinity would be the same as those calculated in this paper.

The author is greatly indebted to Professor L. Infeld for his kind interest in this work. Thanks are also due to Dr A. Trautman for some illuminating discussions.

MORE DETAILED CALCULATIONS WILL BE PUBLISHED IN ACTA PHYS. POLON.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VIII, No. 4, 1960

PHYSIQUE EXPÉRIMENTALE

Sur une méthode permettant de mesurer la répartition de la température sur la surface des cathodes thermioniques planes de très petites dimensions

par

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Présenté par A. JABŁOŃSKI le 21 janvier 1960

Introduction

De manière générale, la mesure de la répartition de la température sur les surfaces des cathodes thermioniques planes ne présente pas de difficultés tant que celles-ci sont de dimensions suffisamment grandes. Pour mesurer la température, on se sert des thermocouples ou d'un pyromètre optique [1]. Cependant, cette méthode d'agir n'est plus valable lorsque les dimensions des cathodes sont très petites.

Le présent travail donne une méthode permettant de mesurer la répartition de la température sur des surfaces planes dans la gamme des températures entre 700° et 900°C.

La méthode consiste à comparer l'intensité de la lumière émise par les éléments particuliers de la surface de la thermocathode avec un élément-étalon. L'image nette et agrandie de la cathode est projetée sur la fente d'un dispositif photoélectrique de mesure de l'intensité de lumière. Si l'on connaît l'agrandissement de l'image de la cathode et les dimensions de la fente, on peut calculer la grandeur de l'élément de la surface, agissant sur le recépteur photoélectrique.

Les déplacements de la cathode permettent de mesurer l'intensité de la lumière émise par les divers éléments de leur surface.

Etalonnage du dispositif

Afin de reporter les indications du dispositif aux différences de température entre les différents éléments de la surface de la cathode, le dispositif photoélectrique fut étalonné de la manière suivante. Deux cathodes en nickel, planes et identiques, à chauffage indirect, furent montées dans des tubes aux fenêtres plan-parallèles, qui furent decellés sous vide. Pour chacun d'eux la température moyenne en fonction de la

puissance du courant de chauffage fut déterminée avec un pyromètre optique. La Fig. 1 présente la disposition des cathodes et des systèmes optiques dans le procédé d'étalonnage du dispositif de mesure de l'intensité de la lumière. On réproduit les images des cathodes, en grandeur

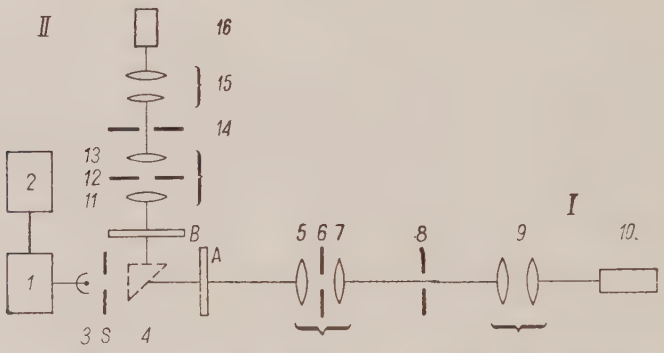

Fig. 1. Système optique pour étalonnage du dispositif de measure de l'intensité lumineuse.
1 — électromètre unifilaire, 2 — compensateur, 3 — cellule photoélectrique, 4 — prisme, 5, 7, 9, 11, 13, 15 — lentilles, 8, 14 — diaphragmes, 6, 12 — diaphragmes à réglage, 10, 16 — tubes contenant les cathodes, S — fente, A, B — diaphragmes

naturelle sur la fente dont l'ouverture est suffisamment large pour laisser passer le faisceau entier. Un prisme à réflexion totale, disposé devant la fente, permet d'y projeter la lumière émise soit par l'une soit

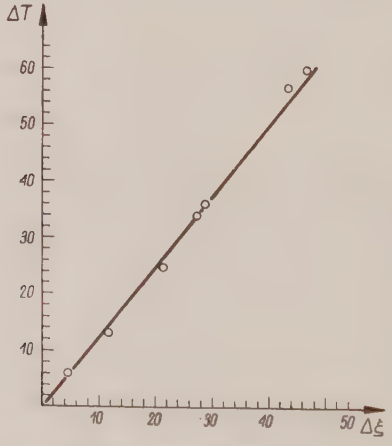

Fig. 2. Indications du dispositif de mesure photoélectrique en fonction des différences des températures

par l'autre des deux cathodes. La puissance de chauffage des cathodes est ajustée de façon à rendre leurs températures moyennes égales. Ensuite, en faisant usage des diaphragmes (6 et 12, Fig. 1), on rend égales les intensités des faisceaux lumineux provenant de chacune des deux ca-

thodes. En changeant la température moyenne d'une des cathodes, tenant une autre constante, on peut tracer une courbe d'étalonnage: la différence de la température des cathodes comparées contre les indications du dispositif photoélectrique.

Les auteurs ont utilisé le dispositif photoélectrique de mesure de l'intensité employé dans le spectrophotomètre Zeiss, composé d'une photocellule, d'un électromètre unifilaire et d'un compensateur. La Fig. 2 présente les indications de l'appareil en fonction des différences des températures des cathodes, pour une photocellule Zeiss du type MV.

Pour l'intervalle entre 700° et 900°C on obtient une ligne droite. Leur coefficient angulaire est égal à 1,28.

Mesures de la distribution de la température

Dans les mesures de la distribution de la température, on fait usage d'un système optique donnant, sur la fente, une image fortement agrandie (15—30 fois) de la cathode. L'autre cathode est remplacée par une source lumineuse stabilisée qui sert d'étalon (une lampe incadescente à fil de tungstène). La température de la source-étalon doit être rapprochée ou égale à la température moyenne mesurée de la cathode et doit être comprise entre 700° et 900°C. La source-étalon sert en même temps à contrôler le fonctionnement du dispositif de mesure.

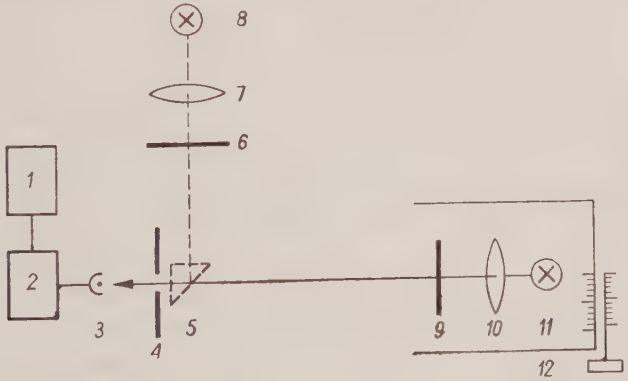

Fig. 3. Diagramme du dispositif pour mesurer la répartition de la température
1 — compensateur, 2 — électromètre unifilaire, 3 — cellule photoélectrique, 4 — fente, 5 — prisme, 6, 7, 9, 10 — objectifs, 8 — source-étalon, 11 — cathode examinée, montée dans un tube, 12 — support micrométrique

Par un réglage approprié des dimensions des diaphragmes (6 et 9, Fig. 3) on a pu égaliser l'intensité de la lumière étalon à celle de la lumière émise par un élément arbitraire de la surface de la cathode. Ainsi, à l'aide du diagramme de la Fig. 2, on obtient la distribution des températures par rapport à un élément arbitraire de la surface de la cathode, en admettant que leurs pouvoirs émissifs sont très rapprochés

[2]. Les différences des températures peuvent être déterminées à $\pm 0.13°C$ près pour un élément de la surface de la cathode égale à 720 μ^2 et à une température moyenne de cathode en nickel de 700°C. La Fig. 3 présente la disposition de l'appareil pour mesurer la distribution des températures.

Application

La Fig. 4 représente la courbe de la distribution de la température le long du diamètre de la cathode. Celle-ci, à surface plane (d'un diamètre de 1 mm), était en nickel. Elle fut montée sur un support fixé d'un côté de la cathode. La courbe montre le gradient de température résultant de ce montage.

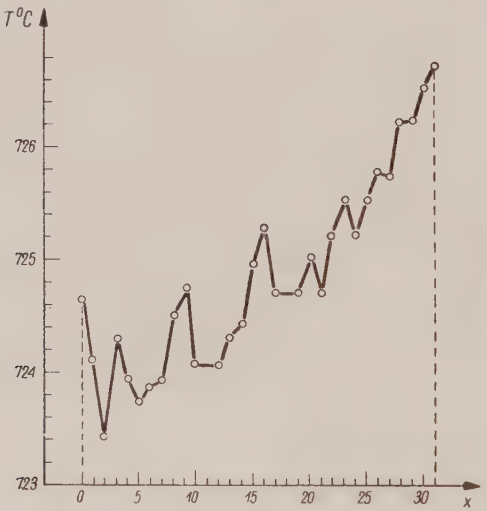

Fig. 4. Répartition de la température le long du diamètre d'une cathode plane en nickel

Au cours de ces mesures, les dimensions de la fente étaient de $0.2 \times \times 2$ mm. L'agrandissement de l'image était de 15 fois. Après chaque mesure, les lampes furent déplacées de 0.5 mm sur le support micrométrique.

Les auteurs remercient M. M. les Professeurs A. Jabłoński et T. Skaliński de leurs conseils et discussions précieux, ainsi que M. le Directeur W. Barwicz d'avoir donné son agrément à la publication du présent travail.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VIII, No. 4, 1960

EXPERIMENTAL PHYSICS

On the Notion of Emission Anisotropy

by

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Presented on February 12, 1960

The aim of this paper is to show how much simpler become several expressions involving polarization of photoluminescence, if the notion of emission anisotropy r is used, instead of that of degree (or rate) of polarization P, or of that of degree of depolarization ϱ. The notion of emission anisotropy was introduced in a paper on decay of photoluminescence [8], and since then used by several authors [1]—[4], [9], [10].

Emission anisotropy, in the case of excitation of an isotropic luminophor by plane polarized light, is defined as:

$$r_p = \frac{I^{\parallel} - I^{\perp}}{I^{\parallel} + 2I^{\perp}} = \frac{I^{\parallel} - I^{\perp}}{I}, \tag{1}$$

where $I^{\parallel}$ denotes the intensity of the component of photoluminescence parallel to the electric vector of the primary light, $I^{\perp}$ that perpendiculuar to it, and $I = I^{\parallel} + 2I^{\perp}$ the sum of intensities of all three components *). Symmetry of the problem (the direction of $I^{\parallel}$ being its symmetry axis) requires both $I^{\perp}$ to be equal.

From (1) it follows that

$$I^{\parallel} = \frac{I}{3}(1 + 2r_p) \tag{2}$$

and

$$I^{\perp} = \frac{I}{3}(1 - r_p). \tag{3}$$

In the case of excitation of an isotropic luminophor by natural light the symmetry axis of the problem is directed along the exciting beam. The two components $I_{\parallel}$ of photoluminescence perpendicular to this direc-

*) It is easy to show that $r_p = \frac{2P_p}{3 - P_p} = \frac{1 - \varrho_p}{1 + 2\varrho_p}$, $P_p = \frac{3r_p}{2 + r_p}$ and $\varrho_p = \frac{1 - r_p}{1 + 2r_p}$, where $P_p = \frac{I^{\parallel} - I^{\perp}}{I^{\parallel} + I^{\perp}}$ is the degree of polarization and $\varrho_p = \frac{I^{\perp}}{I}$ the degree of depolarization.

tion are equal, and I_2, parallel to it, is, in general, different from I_1 The emission anisotropy r_n in this case is defined as

$$r_n = \frac{I_1 - I_2}{2I_1 + I_2} = \frac{I_1 - I_2}{I}, \tag{4}$$

whence

$$I_1 = \frac{I}{3}(1 + r_n) \tag{5}$$

and

$$I_2 = \frac{I}{3}(1 - 2r_n). \tag{6}$$

Since $I_1 = I^{\parallel} + I^{\perp}$ and $I_2 = 2I^{\perp}$, r_n is simply:

$$r_n = \frac{r_p}{2}, \quad \text{or} \quad r_p = 2r_n. \tag{7}$$

The corresponding expressions for degree of polarization P_n (cf. [4]), and those for degree of depolarization ϱ_n are

$$P_n = \frac{P_p}{2 - P_p}, \quad P_p = \frac{2P_n}{1 - P_n}, \quad \varrho_n = \frac{2\varrho_p}{1 + \varrho_p} \quad \text{and} \quad \varrho_p = \frac{\varrho_n}{2 - \varrho_n}.$$

The expressions for fundamental polarization in the general case of different absorption and emission oscillators (cf. [5], [6]) are rather involved, and will not be quoted here. The corresponding expression for fundamental emission anisotropy is simply

$$r_0 = 0.6 \sum_{i=1}^{3} \Gamma_i G_i - 0.2, \tag{8}$$

where Γ_1, Γ_2, Γ_3 denote the relative values of principal transition probabilities of the "absorption oscillator", and G_1, G_2, G_3 — the components of the transition probabilities of the "emission oscillator" along the directions of principal axes of the absorption oscillator (the relative principal transition probabilities must be normalized so, as to give $\Gamma_1 + \Gamma_2 + \Gamma_3 = 1$, and $G_1 + G_2 + G_3 = 1$).

Eq. (8) reduces to

$$r_0 = 0.6 \cos^2 \alpha - 0.2, \tag{9}$$

if both the absorption and the emission oscillators are linear oscillators making an angle α *).

The limits of possible values of r_0 are $-0.2 \leqslant r_0 \leqslant +0.4$.

If the absorption and emission oscillators are identical, Eq. (8) becomes

$$r_0 = 0{,}6 \sum_{i=1}^{3} \Gamma_i^2 - 0.2. \tag{10}$$

*) The corresponding expression in terms of degree of polarization, given by Perrin [12], reads $P_0 = \frac{3\cos^2 \alpha - 1}{\cos^2 \alpha + 3}$.

Expressions for emission anisotropy of molecules executing torsional vibrations are also very much simpler than those for polarization *).

It is worth noting that, as it follows from Eqs. (1) and (4), emission anisotropy of a luminophor containing different groups of centres with different r_i and I_i is

$$r = \frac{\sum_i r_i I_i}{\sum_i I_i}. \tag{11}$$

Eq. (11) is valid for r_p, as well as for r_n.

As last two examples of application of the notion of emission anisotropy the decay of components of polarized photoluminescence and the process of depolarization of photoluminescence will be considered.

If there are depolarizing factors in a luminophor, as is usually the case, the observed absolute value of r is smaller than that of fundamental emission anisotropy r_0. If the process of depolarization lasts a time comparable with the mean duration of luminescence, it affects in general the decay of luminescence: the decay does not in general follow the simple exponential law, and different components of photoluminescence decay in different ways and with different mean durations **). This was shown long ago [5] for the simplest case, in which the depolarization probability φ per unit time (for definition see Eq. (12)) can be considered as constant and the same for all luminescent centres in a luminophor (more general case, in which this condition is not fulfilled was treated in [8]). The expressions then obtained, in terms of the degree of depolarization ϱ, and the way in which they were obtained, are rather involved. Everything becomes simple, if r is used instead of ϱ.

It can be easily shown that, provided the above condition for φ is fulfilled, the momentary value of r decreases from the moment of excitation, according to $dr = -\varphi r\, dt$, whence

$$r = r_0 e^{-\varphi t}, \tag{12}$$

where t is the decay time, and r_0 the fundamental emission anisotropy. Eq. (12) describes the process od depolarization of luminescence produced by Brownian rotations of luminescent molecules in solutions considered

*) As an example an expression may be quoted for polarization of photoluminescence emitted by a linear oscillator executing vibrations about its equilibrium direction (cf. [12], [7]) $P'_p = \frac{9u^2-12u+4}{3u^2-4u+8}$, where u is the dispersion of the sinus of the angle made by the oscillator with its mean direction. The corresponding expression for emission anisotropy is $r'_p = 0.4 - 1.2\,u + 0.9\,u^2$.

**) Some depolarizing factors, as e.g. torsional vibrations of luminescent molecules in solutions, produce a reduction of emission anisotropy practically instantly (cf. [7]). Such factors do not affect the decay process, and will not be taken into account in present considerations.

by Perrin [12] (in this case $\varphi = kT/v\eta$, η being the viscosity coefficient of the solvent, v — the effective volume of the luminescent molecule, and k and T having the usual meanings), and some of depolarization processes in vapours, φ denoting in this case the number of depolarizing collisions per unit time (usually not r is observed, but rather its mean value $\bar{r}$ over the decay time, cf. Eq. (16); special experiments for measuring the momentary value of polarization were performed by Galanin [3]).

Let the decay of the total emitted photoluminescence be given by

$$I = I_0 e^{-\gamma t}. \tag{13}$$

Then, as follows from Eqs. (2), (3), (12) and (13) for the case of excitation with plane polarized primary light,

$$I^{\parallel} = \frac{I_0}{3}(1+2r_0 e^{-\varphi t}) e^{-\gamma t} \tag{14}$$

and

$$I^{\perp} = \frac{I_0}{3}(1-r_0 e^{-\varphi t}) e^{-\gamma t}. \tag{15}$$

The observed emission anisotropy is the mean value of r:

$$\bar{r} = \frac{\int_0^\infty I^{\parallel} dt - \int_0^\infty I^{\perp} dt}{\int_0^\infty I dt} = \frac{\int_0^\infty r I dt}{\int_0^\infty I dt} = \frac{r_0 \int_0^\infty e^{-(\varphi+\gamma)t} dt}{\int_0^\infty e^{-\gamma t} dt} = r_0 \frac{\gamma}{\gamma+\varphi}. \tag{16}$$

Putting $\gamma = 1/\tau$, where τ is the mean life of the excited state of the emitting molecule, we obtain

$$\bar{r} = \frac{r_0}{1+\varphi\tau}, \tag{17}$$

an expression having the same form as the well known Stern-Volmer formula for quenching of photoluminescence.

By means of (16) Eqs. (14) and (15) become

$$I^{\parallel} = \frac{I_0}{3}\left(e^{-\gamma t} + 2r_0 e^{-\gamma \frac{r_0}{\bar{r}}}\right) \tag{18}$$

and

$$I^{\perp} = \frac{I_0}{3}\left(e^{-\gamma t} - r_0 e^{-\gamma \frac{r_0}{\bar{r}}}\right). \tag{19}$$

The mean durations of decay of these components are

$$\tau^{\parallel} = \frac{\int_0^\infty t\, I^{\parallel} dt}{\int_0^\infty I^{\parallel} dt} = \tau \frac{r_0 + 2\bar{r}^2}{r_0 + 2r_0\bar{r}} \tag{20}$$

and

$$\tau^{\perp} = \frac{\int_0^{\infty} t I^{\perp} dt}{\int_0^{\infty} I^{\perp} dt} = \tau \frac{r_0 - r^2}{r_0 - r_0 r}. \tag{21}$$

The above theory (in its original form) was checked experimentally by Szymanowski [13], Kessel [11] and Galanin [3]. The agreement is rather qualitative. The deviations are probably due either to the presence of some depolarizing factors not influencing $\tau^{\parallel}$ and $\tau^{\perp}$ at all (as e.g. torsional vibrations of luminescent molecules), or producing the decrease of r in a way differing from that given by Eq. (12).

It follows from Eqs. (5)—(7) and (13) that in the case of excitation with natural light

$$I_1 = \frac{I_0}{3} (e^{-\gamma t} + r_{n0} e^{-\gamma \frac{r_{n0}}{\bar{r}_n} t}), \tag{22}$$

$$I_2 = \frac{I_0}{3} (e^{-\gamma t} - 2 r_{n0} e^{-\gamma \frac{r_{n0}}{\bar{r}_n} t}), \tag{23}$$

$$\tau_1 = \tau \frac{r_{n0} + \bar{r}_n^2}{r_{n0} + r_{n0} \bar{r}_n}, \tag{24}$$

$$\tau_2 = \tau \frac{r_{n0} - 2 \bar{r}_n^2}{r_{n0} - 2 r_{n0} \bar{r}_n}. \tag{25}$$

As was already shown in [5], there exists a component decaying according to the simple exponential law (Eq. (13)) with a decay constant equal to that of the decay of the total emitted light. In the case of polarized excitation it is the component making an angle $\alpha = \operatorname{arc\,tg} \sqrt{2}$ with $I^{\parallel}$, in the case of excitation with natural light the angle between such a component and I_1 is $\alpha = \operatorname{arc\,tg} \sqrt{0.5}$, if observation is made at a right angle to the exciting beam.

It is worth noting that also in the case, in which two components ($I^{\parallel}$ and $I^{\perp}$, or I_1 and I_2) are simultaneously observed, the decay does not follow in general the simple exponential law.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VIII, No. 4, 1960

EXPERIMENTAL PHYSICS

Influence of Temperature on the Broadening of the Mercury Resonance Line by Xenon

by

T. SKALIŃSKI, J. ROGACZEWSKI and M. I. KAPUŚCIŃSKA

Presented by A. JABŁOŃSKI on February 17, 1960

Introduction

In [1] T. Skaliński has investigated the influence of temperature on the broadening of the mercury resonance line (2537 Å) by argon. It was found that at constant densities of argon and of the mercury vapour, the breadth of the line remains unchanged when the number of atomic collisions is increased (by the change of temperature of the absorption tube within the limits 320°K to 1250°K). This result, being quite in disagreement with the prediction of the collision theories of the pressure broadening, has shown the necessity of investigation of this effect in a more systematic way for the various foreign gases as well as for different spectral lines. The present paper concerns the investigation of the mercury resonance line 2537 Å broadened by xenon.

Experimental

The apparatus in the present experiment was in principle analogous to that of [1] and will not be desribed here in detail. Instead of a hydrogen tube of the "end on" type, a hydrogen tube with stabilized current supply taken from Zeiss-Spectralphotometer was used. The silica absorption tube 1 m in length and 30 mm in diameter was prepared and filled with spectrally pure xenon and mercury in an analogous way as decribed for argon. The xenon pressure at 20°C was $p_{Xe} = 300.1$ mm Hg which corresponds to about 1×10^{19} at/ccm. The concentration of the mercury vapour was 4.2×10^{14} at/ccm. it was sufficient, to ensure total absorption in the centre of the resonance line.

In every measure series the absorption line was photographed on the same plate at seven different furnace temperatures (from about 300°K to 1275°K). Since in this case, contrary to the results obtained for the broadening by argon, the changes in the line shape, as well as in magnitude, of the total absorption were observed with the change of temperature, photometric marks were made on each plate by photographing the background given by the hydrogen tube throughout an eight-step neutral filter, besides the absorption spectra. The blackening distribution curves of the investigated line, obtained by means of the thermoelectric, selfrecording microphotometer (with 100-fold magnification) were then recalculated for true intensity distribution curves. All these curves were plotted in the same size and the total absorption was found by measuring the area enclosed by the absorption line.

Results

The investigations of the temperature influence on the broadening of the mercury resonance line by xenon consisted in measurements of the total absorption, measurements of the line asymmetry, evaluation of the exponent in the formula expressing the intensity distribution in the long wave wing and an attempt of analysis of the band structure revealed on the long wave wing of the line.

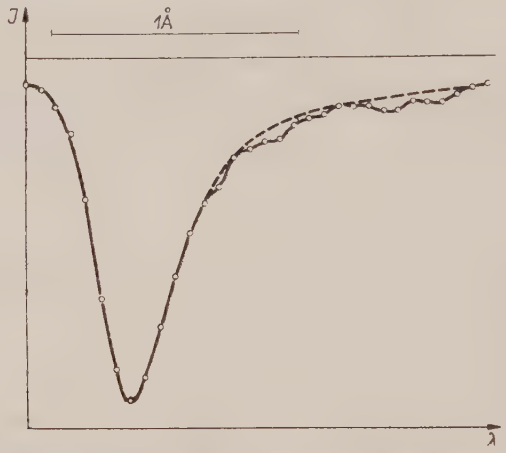

Fig. 1. Shape of the broadened mercury resonance line (broken line: the intensity distribution calculated after the Kuhn formula)

Although the xenon pressure was almost the same as that of argon in the previous experiment the breadth of the line and its asymmetry were much greater when broadened with xenon.

The characteristic shape of the broadened line contour is shown in Fig. 1.

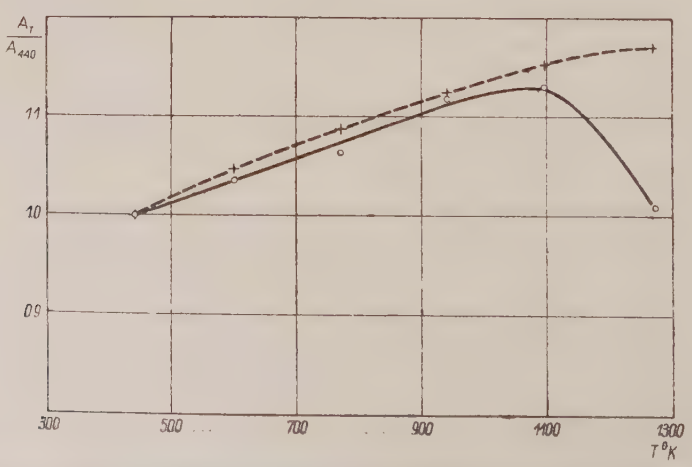

Fig. 2. Ratio of total absorption at high temperature T to that at 400°K *vs* temperature (broken line: the theoretical curve after Lenz-Weisskopff).

The total absorption in the temperature region from 320°K to about 1000°K increases, in apparent agreement with the Lenz-Weisskopff theory [2]. The diagram in Fig. 2 gives the change of the total absorption with temperature variation. The experimental curve (full line) is drawn through several points (each point being a mean value from 13 plates) together with the theoretical curve $A \sim T^{3/20}$ (broken line). On the contrary, in the region of temperatures higher than 1000°K the total absorption rapidly decreases reaching at 1275°K the same value as at 400°K. Thus, within this second temperature region the observed variation of the total absorption is completely diverging from those indicated by the Lenz-Weisskopff theory. This effect seems to be established with certainty. It revealed itself in all measurements. Thus, the general shape of the curve in the entire investigated temperature region shows that the effect of temperature variation of the line broadening cannot be satisfactorily explained by the Lenz-Weisskopff theory.

The line asymmetry A_r/A_v defined as the ratio of the area A_r under the long wave part of the line contour to that of A_v under the short wave part showed a pronounced variation with temperature (Fig. 3; full line). The asymmetry having at 400°K the value of about 2.2 diminishes when the temperature is raised and attains the value 1.8 (at 1275°K).

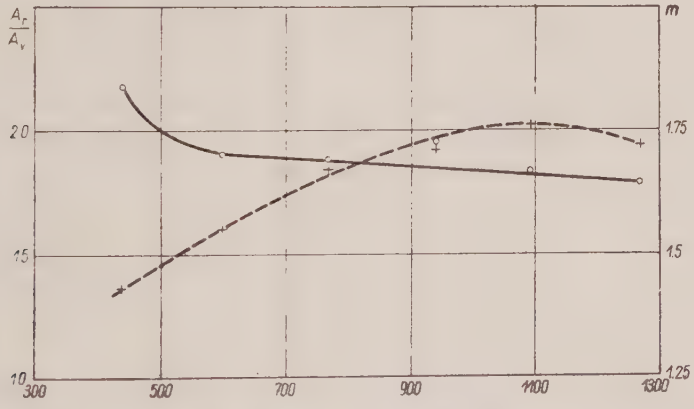

Fig. 3. The change of the asymmetry with temperature (full line) and the change of the exponent in the Kuhn formula with temperature (broken line)

The third, rather interesting effect observed is the temperature dependence of the exponent in the expression describing the intensity distribution in the long wave wing of the broadened line (known as Kuhn formula). Assuming the interaction between an absorbing atom and a perturbing one to be of the van der Waals type (interaction potential proportional to r^{-6}) Kuhn [3] deduced a formula:

$$I(\nu) \sim (\nu_0 - \nu)^{-m},$$

where $m = 3/2$.

Our measurements showed that the exponent m is not constant but changes with temperature from $m=1.4$ to $m=1.75$ (with a flat maximum at about 1100°K, Fig. 3, broken line).

It is to be noticed that Rühmkorff [4], analysing partially his own results and partially the results of the earlier measurements of Füchtbauer, Joos a. Dinkelacker [5] on the broadening of the mercury resonance line by foreign gases (under a pressure much greater than used in the present experiment) found for m the values enclosed within the limits $3/2 \leqslant m \leqslant 2$. For the same foreign gas but for different parts of the line contour (corresponding to different distances from the line centre) he found for m two distinct values. This effect was not observed in our measurements (perhaps owing to the fact that the foreign gas pressure was relatively small and therefore the line broadening not so great as observed by Rühmkorff).

On the long wave wing of the broadened line a few successive maxima and minima are to be seen (cf. Fig. 1, where the broken line represents the intensity distribution calculated after the Kuhn formula). This structure may be interpreted as a system of bands belonging to the Hg-Xe molecule.

In the fluorescence (and partially in absorption) of the mercury vapour mixed with a foreign gas under a high pressure similar bands were observed in the neighbourhood of the broadened line [6].

With krypton and argon these bands show a pronounced vibrational structure. The observed maxima are situated (for the system Hg+Ar) at about 81, 112, 142 and 173 cm^{-1} from the intensity maximum. Using xenon as a foreign gas Oldenberg found in fluorescence only one diffuse maximum on the short wave side of the line.

Fig. 1 shows that the structure observed in our experiment lies much closer to the centre of the line, the mean distances of the observed maxima from the intensity maximum of the line being equal to 5, 8, 10, 12, 14, 16, 18 and 20 cm^{-1}.

If, as usually, we assume that in the excited resonance state the interaction between the Hg* and Xe atoms is represented by two potential curves, with minima of the depth much greater than the one in the fundamental state, the appearence of two systems of bands is to be expected. Nevertheless in this case the overlapping of these bands cannot satisfactorily explain the observed frequency differences, much smaller in comparison with those observed elsewhere for Hg*+Ar or Hg*+Kr [6].

The observed increase of the total absorption in the mentioned temperature region may be caused by the brodening of the vibrational bands superposed on the atomic line of mercury. This effect may be attributed to the changes with temperature in the population of the different rotational states. Since at the highest temperatures attained we have observed

that the band structure disappears (as a consequence of the dissociation of the molecules), the absorption by the molecules vanishes and the resulting absorption is due to the mercury atoms only.

The authors wish to thank Professor A. Jabłoński for many helpful discussions.

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БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ
И ФИЗИЧЕСКИХ НАУК

1960

ТОМ VIII Резюме статей ВЫПУСК 4

Г. РАДЗИКОВСКИЙ, **ПРОБЛЕМА ОДНОЗНАЧНОСТИ ГРАНИЧНОЙ ЗАДАЧИ ДЛЯ УЛЬТРАГИПЕРБОЛИЧЕСКОГО УРАВНЕНИЯ** . . стр. 203—207

В настоящей работе рассматривается проблема однозначности граничной задачи для ультрагиперболического уравнения вида:

$$\Delta_2 u \equiv \sum_{i=1}^{k} g_{ii}(x) \frac{\partial^2 u}{\partial x_i^2} - \sum_{j=k+1}^{k+n} g_{jj}(x) \frac{\partial^2 u}{\partial x_j^2} = 0 , \tag{1}$$

где $g_{ii}(x)$ для $i = 1, 2, \ldots, k+n$ являются коэффициентами метрической формы, имеющими непрерывные частные производные.

Настоящая работа является модификацией метода Л. Гёрмандера, касающегося проблемы однозначности граничных задач для нормальных гиперболических уравнений.

Проблема однозначности решения ультрагиперболического уравнения вида

$$L[u] \equiv \sum_{i=1}^{k} \frac{\partial^2 u}{\partial x_i^2} - \sum_{j=k+1}^{k+n} \frac{\partial^2 u}{\partial x_j^2} = 0 \tag{2}$$

в единичном шаре K была исследована Овенсом. Полученные в работе результаты являются обобщением результатов, полученных Овенсом.

Б. ЛАВРУК, **О ДВУХ ЭЛЛИПТИЧЕСКИХ СИСТЕМАХ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ВТОРОГО ПОРЯДКА** . . . стр. 209—216

Рассматриваются граничные задачи с косой производной в граничном условии для системы А. В. Бицадзе и системы двух уравнений Лапласа. Для последней граничная задача в пространстве, приводимая к регулярным интегральным уравнениям по Я. Б. Лопатинскому, имеет всегда индекс равный нулю.

З. ЦЕСЕЛЬСКИЙ, **ОБ ИЗОМОРФИЗМАХ ПРОСТРАНСТВ H_α И m** стр. 217—222

Обозначим через $\langle m, \|\cdot\| \rangle$ и $\langle c_0, \|\cdot\| \rangle$ соответственно пространства Банаха всех ограниченных и всех стремящихся к нулю действительных последовательностей с нормой определенной как супремум; $\langle H_\alpha, \|\cdot\|_\alpha \rangle$ и $\langle H_{\alpha,0}, \|\cdot\|_\alpha \rangle$ обозначают пространства всех действительных функций, определенных на сегменте $\langle 0,1 \rangle$, исчезающих в нуле, для которых выполнены соответственно условия (1) и (2), а норма $\|\cdot\|_\alpha$ определена формулой (1). Функции $\chi_n^{(\alpha)}$ и $\varphi_n^{(\alpha)}$ определены формулами (3), (4) и (5). Главным результатом является следующая теорема.

Пусть $0 < \alpha < 1$. *Пространства* $\langle H_\alpha, \|\cdot\|_\alpha \rangle$, $\langle H_{\alpha,0}, \|\cdot\|_\alpha \rangle$ *соответственно изоморфны с пространствами* $\langle m, \|\cdot\| \rangle$, $\langle c_0, \|\cdot\| \rangle$. *Изоморфизмы* T_α, $T_{\alpha,0}$ *с* $\langle m, \|\cdot\| \rangle$, $\langle c_0, \|\cdot\| \rangle$ *соответственно на* $\langle H_\alpha, \|\cdot\|_\alpha \rangle$, $\langle H_{\alpha,0}, \|\cdot\|_\alpha \rangle$ *и изоморфизмы* T_α^{-1}, $T_{\alpha,0}^{-1}$ *даны формулами* (6) и (7), *а кроме того, выполнены формулы* (8) *и* (17). *Система* $\{\varphi_n^{(\alpha)}\}$, $n = 1, 2, \ldots$, *является базисом Шаудера пространства* $H_{\alpha,0}$, $\|\cdot\|_\alpha$. В Теореме 3 дается общий вид линейного функционала на $\langle H_{\alpha,0}, \|\cdot\|_\alpha \rangle$.

А. СУЛИНЬСКИЙ, **О ПОДПРЯМЫХ СУММАХ ПРОСТЫХ КОЛЕЦ С ЕДИНИЦЕЙ** . . . . . стр. 223—228

В работе рассматриваются кольца полупростые в смысле радикала Брауна и Маккоя, т. е. подпрямые суммы простых колец с единицей.

Пусть $\mathfrak{r}$ — полупростое кольцо. Тогда множество M всех его максимальных модулярных идеалов называется структурным пространством кольца $\mathfrak{r}$. В пространстве M вводится Стоуновская топология. Минимальным множеством D полупростого кольца $\mathfrak{r}$ называется пересечение всех таких $N \subseteq M$, что $\bar{N} = M$, где N есть замыкание множества N в пространстве M.

Через $I(D)$ обозначим пересечение всех идеалов из D. Если $I(D) = 0$, то кольцо $\mathfrak{r}$ называется специальным, если же $I(D) = \mathfrak{r}$, то кольцо $\mathfrak{r}$ называется вполне неспециальным. Доказываются необходимые и достаточные условия как специальности, так и полной неспециальности полупростого кольца $\mathfrak{r}$. Кроме того, в любом полупростом кольце выделяются его специальная часть и часть вполне неспециальная.

В. КРУЛИКОВСКИЙ, **О ВОЗМОЖНОСТИ МЕХАНИЗМА МАКСИМУМОВ ДЛЯ ВЫСОКОЙ ЭНЕРГИИ В ЭФФЕКТИВНОМ СЕЧЕНИИ НА РАССЕЯНИЕ МЕЗОН π-НУКЛОН** . . . . . стр. 229—231

Движение кора внутри нуклона предлагается как возможная причина трех максимумов для высокой энергии, наблюдаемых в рассеянии $\pi^- p$ и $\pi^+ p$.

Р. МИХАЛЬСКАЯ, **ЗАМЕТКА О ЗАКОНЕ СОХРАНЕНИЯ МОМЕНТА КОЛИЧЕСТВА ДВИЖЕНИЯ СИСТЕМЫ ВРАЩАЮЩИХСЯ ТЕЛ В ОБЩЕЙ ТЕОРИИ ОТНОСИТЕЛЬНОСТИ** . . . . . стр. 233—236

Работа тесно связана с методом Тульчиева выведения уравнений движения вращающихся тел в общей теории относительности.

В настоящей работе выведены уравнения для внутренних моментов количества движения тел; доказывается, что полный момент количества движения, построенный из орбитального момента и внутренних моментов тел, является константой движения.

Р. МИХАЛЬСКАЯ, **ВАРИАЦИОННЫЙ ПРИНЦИП ДЛЯ УРАВНЕНИЙ ДВИЖЕНИЯ ВРАЩАЮЩИХСЯ ТЕЛ В ОБЩЕЙ ТЕОРИИ ОТНОСИТЕЛЬНОСТИ** стр. 237—246

Исходя из фоккеровского интеграла действия для идеальной жидкости, данного Бажанским и Плебанским, в работе выведена функция Лагранжа для системы вращающихся тел в общей теории относительности. Предполагается, что размеры тел невелики по сравнению с расстояниями между ними. Такое предположение позволило определить давление и формы вращающихся тел.

Была произведена ренормализация плотности массы тел и получена функция Лагранжа в сравнительно простом виде. Эта функция описывает сжатые тела в виде эллипсоидов вращения. В ней содержатся выражения линейные в угловых скоростях, отвечающие движению тел сферически симметрических, а также выражения квадратичные по отношению к угловой скорости, связанные со сжатием тел.

В функции Лагранжа содержатся, кроме системы мультипольных моментов тел, другие константы, зависящие от структуры тел. Эти константы указывают на то, что гравитационный скелет в смысле Матиссона, в модели вращающихся жидких тел, не существует.

Р. МИХАЛЬСКАЯ, **УРАВНЕНИЯ ДВИЖЕНИЯ ВРАЩАЮЩИХСЯ СЖАТЫХ ТЕЛ В ОБЩЕЙ ТЕОРИИ ОТНОСИТЕЛЬНОСТИ** . . . . стр. 247—253

Используя функцию Лагранжа, выведенную в предыдущей работе, даются уравнения поступательного и внутреннего движения для системы вращающихся тел в общей теории относительности. Сформулирован закон сохранения полного момента количества движения системы.

В работе вычислены вековые возмущения оскулирующих элементов ньютоновой орбиты, вызванные вращательным движением тел. Получены вековые изменения долготы восходящего узла $\Delta\Omega$ и углового расстояния периастрона от узла $\Delta\omega$ линейно зависящие от угловой скорости, а также вековые изменения наклонения орбиты i квадратично зависящие от угловых скоростей. Изменения $\Delta\Omega$ и $\Delta\omega$ согласуются с результатами, полученными Тульчиевым в продискутированных в его работе особых случаях движения.

М. ФРОНЦКОВЯК и А. ТАЧАНОВСКИЙ, **МЕТОД ИЗМЕРЕНИЙ РАСПРЕДЕЛЕНИЯ ТЕМПЕРАТУРЫ НА ПОВЕРХНОСТЯХ ПЛОСКИХ ТЕРМОЭЛЕКТРИЧЕСКИХ КАТОДОВ ОЧЕНЬ МАЛЫХ РАЗМЕРОВ** . . . стр. 255—258

Разработан метод измерений распределения температуры на плоских поверхностях диаметром менее 1 *мм*. Различия температур в пределах средних температур от 700—900° С измерены на никелевой поверхности с точностью ± 0,13°С. Минимальная поверхность, температура которой измерялась, равняется 720 μ^2.

А. ЯБЛОНСКИЙ, **О ПОНЯТИИ АНИЗОТРОПИИ ЭМИССИИ** . стр. 259—264

Цель настоящей работы состоит в том, чтобы показать, как упрощаются различные выражения, связанные с поляризацией фотолюминесценции, если вместо обычно употребляемой степени поляризации (или степени деполяризации) ввести понятие анизотропии эмиссии. В качестве примеров рассмотрены предельная поляризация и деполяризация люминесценции, затухание поляризованной люминесценции и т. п.

Т. СКАЛИНЬСКИЙ, Ю. РОГАЧЕВСКИЙ и М. КАПУСЬЦИНСКАЯ, **ВЛИЯНИЕ ТЕМПЕРАТУРЫ НА РАСШИРЕНИЕ РЕЗОНАНСНОЙ ЛИНИИ РТУТИ ДАВЛЕНИЕМ КСЕНОНА**. . . . . стр. 265—269

Исследовано влияние числа столкновений на интегральную абсорбцию резонансной линии ртути расширенной ксеноном (плотности ксенона и паров ртути постоянные и равные $1 \cdot 10^{19}$ *ам*/*см*^{3} для ксенона и $4{,}2 \cdot 10^{14}$ *ам*/*см*^{3} для ртути). Число столкновений регулировалось изменением температуры абсорбционной трубы от 300°К до 1275°К.

Измерения по методу фотографической спектроскопии показали, что:

1) в области температур 400°—1000°К интегральная абсорбция возрастает приблизительно в согласии с предложением теории Ленца-Вейскопфа. Однако, при дальнейшем повышении температуры абсорбция понижается и при 1275°К принимает то же самое значение как при 400°К (рис. 2);

2) асимметрия линии падает с повышением температуры (рис. 3 — сплошная кривая);

3) показатель в формуле Куна зависит от температуры (рис. 3 — пунктирная кривая);

4) наблюдается система осцилляционных полос, принадлежащих молекуле Hg*-Xe (рис. 1).

TABLE DES MATIÈRES

Mathématique

1. J. Radzikowski, On the Uniqueness of the Limit Problem for the Ultrahyperbolic Equation 203
2. B. Ławruk, On Two Elliptic Systems of Second Order Linear Differential Equations 209
3. Z. Ciesielski, On the Isomorphism of the Spaces H_a and m . . 217
4. A. Suliński, On Subdirect Sums of Simple Rings with Unity . . . 223

Physique Théorique

5. W. Królikowski, A Possible Mechanism of the High Energy Maxima in Pion-Nucleon Cross Sections 229
6. R. Michalska, On the Conservation Law of the Angular Momentum of a System of Rotating Bodies in the General Theory of Relativity . . . 233
7. R. Michalska, Action Principle for the Motion of Rotating Bodies in the General Theory of Relativity 237
8. R. Michalska, The Equations of Motion of Rotating Oblate Bodies in the General Theory of Relativity 247

Physique Expérimentale

9. M. Frąckowiak, et A. Taczanowski, Sur une méthode permettant de mesurer la répartition de la température sur la surface des cathodes thermioniques planes de très petites dimensions 255
10. A. Jabłoński, On the Notion of Emission Anisotropy 259
11. T. Skaliński, J. Rogaczewski and M. I. Kapuścińska, Influence of Temperature on the Broadening of the Mercury Resonance Line by Xenon 265

Cena zł 20.—